A differential geometric study on strongly pseudo-convex manifolds

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LECTURES IN MATHEMATICS

Department of Mathematics
KYOTO UNIVERSITY

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A DIFFERENTIAL GEOMETRIC STUDY ON STRONGLY PSEUDO-CONVEX MANIFOLDS

BY

NOBORU TANAKA

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CONTENTS

Introduction

Let M be a compact, strongly pseudo-convex (s.p.c.), real hypersurface of a complex manifold M'. By Lewy [16] there is defined on M the (tangential) Cauchy-Riemann operator d''. The operator d'' can be extended to yield the (boundary) complex $\{C^{p,q}(M), d''\}$ due to Kohn-Rossi [15]. Furthermore we have the notion of (tangentially) holomorphic k-forms on M, thus obtaining the holomorphic de Rham complex $\{S^k(M), d\}$. Let $H^{p,q}(M)$ (resp. $H_0^k(M)$) be the cohomology groups of the complex $\{C^{p,q}(M), d''\}$ (resp. of $\{S^k(M), d\}$).

The main purpose of the present note[1) and 2)] is first to make a differential geometric study on the cohomology groups $H^{p,q}(M)$ and

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1) Some of the results in this note have been already announced in the P.J.A. note [31].

2) This note is based on the lectures given at Kyoto University, Nagoya University and Tokyo University, during the years 1971-74.

$H_0^k(M)$, based on the harmonic theory for the complex $\{C^{p,q}(M), d''\}$ which was developed by Kohn [13], and second to try to clarify the the meaning of the cohomology groups in connection with the study of isolated singular points of analytic spaces.

Let us now proceed to the descriptions of the various sections and explain the main results in this note.

§1 is preliminary to the subsequent sections. Let M be a partially complex manifold, which is the abstract model of a real submanifold of a complex manifold. We first introduce the notion of a holomorphic vector bundle E over M and define the associated complex

$$C^0(M, E) \xrightarrow{\bar{\partial}_E} C^1(M, E) \longrightarrow \cdots \longrightarrow C^q(M, E) \xrightarrow{\bar{\partial}_E} \cdots,$$

where the starting operator is the Cauchy-Riemann operator of E. We then introduce a relevant filtration of the de Rham complex of M and let $E_r^{p,q}(M)$ be the associated spectral sequence. The cohomology groups $E_1^{p,q}(M)$ and $E_2^{k,0}(M)$ are of particular importance, which we denote by $H^{p,q}(M)$ and $H_0^k(M)$ respectively. In §2 we specialize a partially complex manifold M to define the notion of a s.p.c. manifold, which is the abstract model of a s.p.c. real hypersurface of a complex manifold. Note that the cohomology groups $H^{p,q}(M)$ and $H_0^k(M)$ mentioned at the outset may be defined in the manner as above. In §§ 3 and 4 the canonical affine connections ∇ of a s.p.c. manifold M and the canonical connections D of a holomorphic vector bundle E over M are discussed.

§§ 5 and 6 are devoted to the analysis (the harmonic theory) for the complex $\{C^q(M, E), \bar{\partial}_E\}$, where E is a holomorphic vector bundle over a compact s.p.c. manifold M. We first describe the Laplacian $\square_E$ in the harmonic theory, in terms of the covariant differentiation D which is induced from the canonical connections ∇ and D stated above (Theorem 5.2). We then prove subellipticity for $\square_E$ by utilizing Theorem 5.2 and state the main theorem essentially due to Kohn [13]. In §7 we apply the general harmonic theory to the complex $\{C^{p,q}(M), d''\}$ and prove a duality theorem (Theorem 7.3) on the cohomology groups $H^{p,q}(M)$. In §8 it is shown that the holomorphic de Rham cohomology groups $H_0^k(M)$ are finite dimensional for all integers k. The proof of this fact is based on the analysis for certain cohomology groups $H_*^{k-1,1}(M)$ (Theorem 8.5). In §9 we study the properties of a differentiable family $\{M_t\}_{t\in\Omega}$ of compact s.p.c. manifolds. Above all we prove a stability theorem (Theorem 9.4) for holomorphic imbeddings of the s.p.c. manifolds M_t in $\mathbb{C}^N$.

In §10 we discuss isolated singular points of complex hypersurfaces Let f be a polynomial function on $\mathbb{C}^{n+1}$, $n \geq 3$, such that $f(0) = 0$ and such that the origin 0 is an isolated critical point of f. Consider the intersection $M = f^{-1}(0) \cap S_\varepsilon$ of the complex hypersurface $f^{-1}(0)$ with the ε-sphere S_ε in $\mathbb{C}^{n+1}$ centred at the origin. For ε sufficiently small, M is proved to be a compact, s.p.c., real hypersurface of $f^{-1}(0)$. Let μ be the Milnor number of the isolated singular point, the origin, of the complex hypersurface. Then we prove the inequality

$$0 < \mu \leq \dim H^{n-1,1}(M) + \dim H_0^n(M)$$

(Theorem 10.5). Thus we see that the singularity has a considerable influence upon the cohomology groups $H^{p,q}(M)$ and $H_0^k(M)$. (Note that if the origin is not a critical point of f, then $H^{p,q}(M) = H_0^k(M) = 0$ for $q \neq 0$, $n - 1$ and for all k (cf. Theorem 10.3 and Milnor [18]).) It is expected that the method of our proof of Theorem 10.5 will be applicable to more general types of isolated singularities. For the Milnor number μ, see also Remark at the end of the section.

§§ 11, 12 and 13 are concerned with the study of normal s.p.c. manifolds. A s.p.c. manifold M is said to be normal if it admits an infinitesimal automorphism ξ satisfying a certain generality condition (an analytic basic field). Typical examples of normarl s.p.c. manifolds are Brieskorn varieties and the $U(1)$-principal fibre bundle canonically associated to negative line bundles over compact complex manifolds (see §11). To every compact normal s.p.c. manifold M there is associated a double complex in a generalized sense, $\{\mathcal{B}^{p,q}(M), \partial, \bar{\partial}\}$, in a canonical manner (see §12). Let $\tilde{H}^{p,q}(M)$ be the cohomology groups of the complex $\{\mathcal{B}^{p,q}(M), \bar{\partial}\}$. In §13 we prove a series of reduction theorems for the cohomology groups $H^{p,q}(M)$ and $H_0^k(M)$, describing these groups in terms of the "reduced " cohomology groups $\tilde{H}^{p,q}(M)$ (Theorems 13.1, 13.2, 13.7, 13.14 and the corollaries to them). These results will be useful for the calculations of $H^{p,q}(M)$ and $H_0^k(M)$. Our discussions here have been motivated by Naruki's (unpublished) theorem on a negative line bundle L over a compact complex manifold A. (This theorem describes the Dolbeault cohomology groups $H^{p,q}(L_0)$ of the non-compact complex manifold $L_0 = L$ - (the zero section), in terms of the cohomology groups $H^q(A, \Omega^p(L^m))$, where $\Omega^p(L^m)$ is the sheaf of local holomorphic p-forms on A with values in L^m, the m-th power of L. Compare Naruki [23], in which he himself gives a generalization of this initial result.)

In the proof of subellipticity for the operator $\Box_E$, Kohn's inequality ([2], Theorem 5.4.7) or Hörmander's inequality (Appendix, Theorem 9) plays a fundamental role (see §6). In Appendix we therefore make some observations about linear differential systems, giving a simple and geometric proof of Hörmander's inequality. We also prove a variant of Hörmander's inequality in terms of Hölder norms (Theorem 7) It should be noted that these inequalities have intimate relationships with certain estimations (Theorem 5) for the distance function ρ associated to a generic differential system Φ.

Finally I would like to thank Dr. Naruki for informing me of his result on line bundles, cited above. I am greatly indebted to Mr. Nakajima for his kind help through reading the manuscript.

Preliminary remarks

Throughout this note we shall always assume the differentiability of class C^∞ unless otherwise stated.

Let M be a differentiable manifold. $T(M)$ will denote the tangent bundle of M. $\mathcal{F}(M)$ (resp. $\mathbb{C}\mathcal{F}(M)$) will denote the space of real (resp. complex) valued C^∞ functions on M.

Let E be a real or complex vector bundle over M. Then $\dim E$ will stand for the fibre dimension of E. For a point $p \in M$, E_p will denote the fibre of E at p. Given an open set U of M, $E|U$ will denote the restriction of E to U. $\Gamma(E)$ will denote the space of C^∞ cross sections of E.

As usual we shall use the notations E^*, $\Lambda^k E$ and $S^k E$ to denote the dual bundle of E, the k-th exterior product of E and the k-th symmetric product of E respectively. For any integers k and ℓ, E^k_ℓ will denote the tensor product

$E \otimes \ldots \otimes E \otimes E^* \otimes \ldots \otimes E^*$ (E k times, E^* ℓ times).

Now consider the tensor products $F \otimes E^0_\ell$, F being another vector bundle over M. Then each fibre $(F \otimes E^0_\ell)_p$ of $F \otimes E^0_\ell$ may be identified with the space of ℓ-linear maps of $E_p \times \ldots \times E_p$ (ℓ times) to F_p. For $\varphi \in (F \otimes E^0_\ell)_p$ and $X \in E_p$, we shall denote by $X \lrcorner\, \varphi$ the element of $(F \otimes E^0_{\ell-1})_p$ defined by

$$(X \lrcorner\, \varphi)(X_1, \ldots, X_{\ell-1}) = \varphi(X, X_1, \ldots, X_{\ell-1}),$$

$$X_1, \ldots, X_{\ell-1} \in E_p.$$

Suppose that E is a real vector bundle. Then $\mathbb{C}E$ will denote the complexification of E, and $\mathbb{C}E \ni u \longrightarrow \bar{u} \in \mathbb{C}E$ will denote the conjugation operator with respect to the real part E of $\mathbb{C}E$.

I. Strongly pseudo-complex manifolds

§1. Partially complex manifolds

1.1 Partially complex manifolds. Let M' be a complex manifold, and S' the subbundle of $\mathbb{C}T(M')$ consisting of all (complexified) tangent vectors of type $(1, 0)$ to M'. Then S' satisfies the conditions :

(C. 1) $\quad \mathbb{C}T(M') = S' + \bar{S}'$ (direct sum);

(C. 2) $\quad [\Gamma(S'), \Gamma(S')] \subset \Gamma(S')$.

Let M be a real submanifold of M'. For each $x \in M$, we define a subspace S_x of $\mathbb{C}T(M)_x$ by

$$S_x = S'_x \cap \mathbb{C}T(M)_x,$$

and assume that $\dim_{\mathbb{C}} S_x$ is constant for all $x \in M$. (In the case where $\operatorname{codim} M = 1$, *this assumption is automatically satisfied.* In fact we have $\dim_{\mathbb{C}} S_x = n-1$, where $n = \dim_{\mathbb{C}} M'$.) Then the union $S = \bigcup_x S_x$ forms a subbundle of $\mathbb{C}T(M)$, and by (C. 1) and (C. 2) we have

(PC. 1) $\quad S \cap \bar{S} = 0;$

(PC. 2) $\qquad [\Gamma(S), \Gamma(S)] \subset \Gamma(S)$.

Let M be a real manifold and S a subbundle of $\mathbb{C}T(M)$. Then S is called a partially complex structure (or a pseudo-complex structure in the terminology of [30]) if S satisfies (PC. 1) and (PC. 2). And the manifold M together with the structure S is called a partially complex manifold. Clearly the notion of a partially complex manifold generalizes that of a complex manifold.

Let M be a partially complex manifold. By (PC. 1), there is a unique subbundle P of $T(M)$ such that

$$\mathbb{C}P = S + \bar{S} \quad \text{(direct sum)},$$

i.e., P is the real part of $S + \bar{S}$. Furthermore there is a unique homomorphism $I : P \longrightarrow P$ such that

$$I^2 = -1;$$

$$S = \{ X - \sqrt{-1}\, IX \mid X \in P \},$$

1 denoting the identity : $P \longrightarrow P$. The pair (P, I), thus obtained, will be called the real expression of S.

Let M_i $(i = 1, 2)$ be partially complex manifolds with structures

S_i. A diffeomorphism $\varphi : M_1 \longrightarrow M_2$ is said to be an isomorphism if the differnential $\varphi_* : \mathbb{C}T(M_1) \longrightarrow \mathbb{C}T(M_2)$ sends S_1 onto S_2. The notion of isomorphisms naturally gives rise to the various notions such as automorphisms, infinitesimal automorphisms (or analytic vector fields), etc.

The (local) equivalence problem of partially complex manifolds was completely solved by Tanaka [30], under some natural assumptions. See also Tanaka [27], [28] and [29] .

1.2. Holomorphic vector bundles. Let M be a partially complex manifold with structure S. For $u \in \mathbb{C}F(M)$, we define $d''u \in \Gamma(\bar{S}^*)$ by

$$(d''u)(\bar{X}) = \bar{X}\, u, \qquad X \in S_x.$$

The differential operator $d'' : \mathbb{C}F(M) \ni u \longrightarrow d''u \in \Gamma(\bar{S}^*)$ is called the (tangential) Cauchy-Riemann operator, and a solution u of the equation $d''u = 0$ is called a holomorphic function.

A complex vector bundle E over M is said to be holomorphic if there is given a differential operator

$$\bar{\partial}_E \ : \ \Gamma(E) \longrightarrow \Gamma(E \otimes \bar{S}^*)$$

satisfying the following conditions :

(HV. 1) $\bar{X}(fu) = \bar{X}f \cdot u + f \cdot \bar{X}u$;

(HV. 2) $[\bar{X}, \bar{Y}]u = \bar{X}\bar{Y}u - \bar{Y}\bar{X}u$,

where $u \in \Gamma(E)$, $f \in \mathbb{C}F(M)$, $X, Y \in \Gamma(S)$ and we put $\bar{Z}u = (\bar{\partial}_E u)(\bar{Z})$, $Z \in \Gamma(S)$. The operator $\bar{\partial}_E$ is called the Cauchy-Riemann operator, and a solution u of the equation $\bar{\partial}_E u = 0$ is called a holomorphic cross section. It is clear that the trivial vector bundle $M \times \mathbb{C}$ is holomorphic with respect to the operator d'' defined above.

Remarks. (1) In the case where M is a complex manifold, our definition of a holomorphic vector bundle is equivalent to the usual one in terms of holomorphic transition functions. We can see this fact, for example, by use of Newlander-Nirenberg's theorem [24].

(2) Consider the case where $M \subset M'$, i.e., M is realized as a real submanifold in a complex manifold M'. Let E' be a holomorphic vector bundle over M'. Then the restriction $E = E'|M$

of E' to M is naturally a holomorphic vector bundle : We have $\bar{X}u = \bar{X}u'$ for all $u' \in \Gamma(E')$ and $X \in S$, where u denotes the restriction of u' to M.

As for holomorphic vector bundles, we have the notions such as homomorphisms, isomorphisms, the tensor products, etc, which are all defined in natural manners. For example, let E and F be two holomorphic vector bundles. Then a bundle homomorphism $\varphi : E \longrightarrow F$ is called holomorphic if

$$\bar{X}(\varphi(u)) = \varphi(\bar{X}u), \qquad u \in \Gamma(E), \quad X \in S,$$

and the tensor product $E \otimes F$ becomes a holomorphic vector bundle by the rule :

$$\bar{X}(u \otimes v) = (\bar{X}u) \otimes v + u \otimes (\bar{X}v), \quad u \in \Gamma(E), \quad v \in \Gamma(F), \quad X \in S.$$

We now show that the factor bundle

$$\hat{T}(M) = \mathbb{C}T(M) \,/\, \bar{S}$$

is a holomorphic vector bundle with respect to the operator $\bar{\partial} = \bar{\partial}_{\hat{T}(M)}$ defined as follows : Let $\tilde{\omega}$ be the projection : $\mathbb{C}T(M) \longrightarrow \hat{T}(M)$. Take any $u \in \Gamma(\hat{T}(M))$ and express it as

$u = \tilde{\omega}(Z)$, $Z \in \Gamma(\mathbb{C}T(M))$. For any $X \in \Gamma(S)$, define a cross section $(\bar{\partial}u)(\bar{X})$ of $\hat{T}(M)$ by

$$(\bar{\partial}u)(\bar{X}) = \tilde{\omega}([\bar{X}, Z]).$$

Then we see easily that $(\bar{\partial}u)(\bar{X})$ does not depend on the choice of Z and that $\bar{\partial}u$ gives a cross section of $\hat{T}(M) \otimes \bar{S}^*$. Furthermore we can show that the operator $u \longrightarrow \bar{\partial}u$ satisfies (HV. 1) and (HV. 2), using the Jacobi identity in the Lie algebra $\Gamma(\mathbb{C}T(M))$. The holomorphic vector bundle $\hat{T}(M)$, thus defined, will be called the holomorphic tangent bundle of M.

Remark. Consider the case where $M \subset M'$. First we note that $\hat{T}(M')$ may be regarded as the holomorphic vector bundle S' of tangent vectors of type $(1, 0)$ to M'. Let E be the restriction of $\hat{T}(M')$ to M. Then the natural map : $\mathbb{C}T(M) \longrightarrow \mathbb{C}T(M')$ induces an injective homomorphism of $\hat{T}(M)$ to E as holomorphic vector bundles. Therefore if $\dim_{\mathbb{C}}\hat{T}(M) = \dim_{\mathbb{C}}M'$ (e.g., if codim $M = 1$), then $\hat{T}(M)$ may be identified with E.

1.3. The cohomology groups $H^q(M, E)$. Let E be a holomorphic vector bundle over M. We put

$$C^q(M, E) = E \otimes \Lambda^q \bar{S}^*,$$

$$\mathcal{C}^q(M, E) = \Gamma(C^q(M, E))$$

and define differential operators

$$\bar{\partial}_E^q : \mathcal{C}^q(M, E) \longrightarrow \mathcal{C}^{q+1}(M, E)$$

by $$(\bar{\partial}_E^q \varphi)(\bar{X}_1, \ldots, \bar{X}_{p+1}) = \sum_i (-1)^{i+1} \bar{X}_i(\varphi(\bar{X}_1, \ldots, \hat{\bar{X}}_i, \ldots, \bar{X}_{p+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \varphi([\bar{X}_i, \bar{X}_j], \bar{X}_1, \ldots, \hat{\bar{X}}_i, \ldots, \hat{\bar{X}}_j, \ldots, \bar{X}_{p+1}),$$

for all $\varphi \in \mathcal{C}^q(M, E)$ and $X_1, \ldots, X_{p+1} \in \Gamma(S)$. Just as in the case of the exterior differentiation d, we can show that $\bar{\partial}_E^q \varphi$ gives an element of $\mathcal{C}^{q+1}(M, E)$ and that $\bar{\partial}_E^{q+1} \circ \bar{\partial}_E^q = 0$. Thus the collection $\{ \mathcal{C}^q(M, E), \bar{\partial}_E^q \}$ gives a complex and we denote by $H^q(M, E)$ the cohomology groups of this complex.

1.4. The spectral sequence $\{ E_r^{p,q}(M) \}$. Let $\{ A^k(M), d \}$ be the de Rham complex of M with complex coefficients, and $H^k(M)$ the cohomology groups of this complex, the de Rham cohomology groups.

If we put

$$A^k(M) = \Lambda^k(\mathbb{C}T(M))^*,$$

we have $\mathcal{A}^k(M) = \Gamma(A^k(M))$. For any integers p and k, we denote by $F^p(A^k(M))$ the subbundle of $A^k(M)$ consiting of all $\varphi \in A^k(M)$ which satisfy the equality :

$$\varphi(X_1, \ldots, X_{p-1}, \bar{Y}_1, \ldots, \bar{Y}_{k-p+1}) = 0$$

for all $X_1, \ldots, X_{p-1} \in \mathbb{C}T(M)_x$ and $Y_1, \ldots, Y_{k-p+1} \in S_x$, x being the origin of φ. Then we have

$$F^p(A^k(M)) \supset F^{p+1}(A^k(M)),$$

$$F^0(A^k((M)) = A^k(M) \ , \quad F^{p+1}(A^p(M)) = 0.$$

Furthermore putting

$$F^p(\mathcal{A}^k(M)) = \Gamma(F^p(A^k(M)),$$

we easily find that

$$dF^p(\mathcal{A}^k(M)) \subset F^p(\mathcal{A}^{k+1}(M)).$$

Thus the collection $\{F^p(\mathcal{A}^k(M))\}$ gives a filtration of the de Rham complex. Let $\{E_r^{p,q}(M)\}$ denote the spectral sequence associated with this filtration.

The groups $E_1^{p,q}(M)$ and $E_2^{k,0}(M)$ are of particular importance, which will be denoted by $H^{p,q}(M)$ and $H_0^k(M)$ respectively. We put as follows :

$$\mathcal{A}^{p,q}(M) = F^p(\mathcal{A}^{p+q}(M)), \quad A^{p,q}(M) = \Gamma(\mathcal{A}^{p,q}(M)),$$

$$\mathcal{C}^{p,q}(M) = \mathcal{A}^{p,q}(M) / \mathcal{A}^{p+1,q-1}(M), \quad C^{p,q}(M) = \Gamma(\mathcal{C}^{p,q}(M)).$$

Then the groups $H^{p,q}(M)$ are the cohomology groups of the complex $\{ C^{p,q}(M), d'' \}$, where the operator $d'' : C^{p,q}(M) \longrightarrow C^{p,q+1}(M)$ is naturally induced from the operator $d : A^{p,q}(M) \longrightarrow A^{p,q+1}(M)$.

Now $E^p = \Lambda^p(\hat{T}(M))^*$ is a holomorphic vector bundle by the rule :

$$(\bar{Y}\varphi)(u_1, \ldots, u_p) = \bar{Y}(\varphi(u_1, \ldots, u_p)) + \sum_i (-1)^i \varphi(\bar{Y}u_i, u_1, \ldots, \hat{u}_i, \ldots, u_p),$$

where $\varphi \in \Gamma(E^p)$, $u_1, \ldots, u_p \in \Gamma(\hat{T}(M))$, $Y \in S$ and $\bar{Y}\varphi = (\bar{\partial}_{E^p}\varphi)(\bar{Y})$, $\bar{Y}u_i = (\bar{\partial}_{\hat{T}(M)}u_i)(\bar{Y})$.

Proposition 1.1. $C^{p,q}(M)$ may be identified with $C^q(M, E^p)$ in a natural manner and we have

$$d''\varphi = (-1)^p \bar{\partial}_{E^p}\varphi, \qquad \varphi \in C^{p,q}(M).$$

Proof. Define a map $\iota^p : A^{p,q}(M) \longrightarrow C^q(M, E^p)$ by

$$(\iota^p\varphi)(\tilde{\omega}(X_1), \ldots, \tilde{\omega}(X_p) \; ; \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= \varphi(X_1, \ldots, X_p, \bar{Y}_1, \ldots, Y_q),$$

for all $\varphi \in A^{p,q}(M)_x$, $X_1, \ldots, X_p \in \mathbb{C}T(M)_x$, and $Y_1, \ldots, Y_q \in S_x$. (It is clear that ι^p is well defined.) Then we have the exact sequence of vector bundles :

$$0 \longrightarrow A^{p+1,q-1}(M) \longrightarrow A^{p,q}(M) \xrightarrow{\iota^p} C^q(M, E^p) \longrightarrow 0 ,$$

whence $C^{p,q}(M) \cong C^q(M, E^p)$. Furtheremore we can easily verify the equalities :

$$\bar{\partial}_{E^p} \iota^p \varphi = (-1)^p \iota^p d\varphi = (-1)^p d'' \iota^p \varphi, \quad \varphi \in A^{p,q}(M),$$

proving Proposition 1.1.

The groups $H_0^k(M)$ are the cohomology groups of the complex $\{ S^k(M), d \}$, where we put $S^k(M) = E_1^{k,0}(M)$. Note that $S^k(M)$ may be characterized as the space of holomorphic k-forms, i.e., holomorphic cross sections of E^p. Thus the complex $\{ S^k(M), d \}$ (resp. the groups $H_0^k(M)$) will be called the holomorphic de Rham complex (resp. the holomorphic de Rham cohomology groups).

Since $dA^{p,q}(M) \subset A^{p,q+1}(M)$, the collection $\{ A^{p,q}(M), d \}$ becomes a complex, which is usually known as the complex of "the mapping cone ". We denote by $H_*^{p,q}(M)$ the cohomology groups of this complex.

Proposition 1.2. We have the natural exact sequences of cohomology groups :

$$0 \longrightarrow H_0^k(M) \longrightarrow H_*^{k-1,1}(M) \longrightarrow H^{k-1,1}(M) \longrightarrow H_*^{k,1}(M) \longrightarrow \cdots .$$

Proof. The short exact sequences

$$0 \longrightarrow A^{k,q}(M) \longrightarrow A^{k-1,q+1}(M) \longrightarrow C^{k-1,q+1}(M) \longrightarrow 0$$

induce the exact sequences of cohomology groups :

$$H_*^{k,0}(M) \longrightarrow H_*^{k-1,1}(M) \longrightarrow H^{k-1,1}(M) \longrightarrow H_*^{k,1}(M) \longrightarrow \cdots .$$

We have

$$H_*^{k,0}(M) = \{ \varphi \in S^k(M) \mid d\varphi = 0 \} ,$$

$$H_*^{k,0}(M) \cap dA^{k-1,0}(M) = dS^{k-1}(M) .$$

From these facts follows immediately Proposition 1.2.

Remrks. (1) Consider the case where $M \subset M'$ and $\operatorname{codim} M = 1$. Let $\{ \underline{C}^{p,q}, d''\}$ be the complex in the sheaf category, associated with the complex $\{ C^{p,q}(M), d''\}$. Then it is easy to see that the complex $\{ \underline{C}^{p,q}, d''\}$ coincides with the (boundary) complex

$\{ \underline{\mathcal{B}}^{p,q}, \bar{\partial}_b \}$ introduced by Kohn-Rossi [15], P.465. We note that they erroneously expressed $\underline{\mathcal{B}}^{p,q}$ as the sheaf of germs of local cross sections of $\Lambda^p S^* \otimes \Lambda^q \bar{S}^*$. However we shall be also concerned with a complex $\{\mathcal{B}^{p,q}(M), \bar{\partial}\}$, where $\mathcal{B}^{p,q}(M) = \Gamma(\Lambda^p S^* \otimes \Lambda^q \bar{S}^*)$, under the assumption that M is a normal s.p.c. manifold (see Chapter III), and clarify the intimate relationship existing between the two complexes $\{ C^{p,q}(M), d'' \}$ and $\{ \mathcal{B}^{p,q}(M), \bar{\partial}\}$.

(2) Suppose that M is a complex manifold. Then the groups $H^{p,q}(M)$ (resp. $H_0^k(M)$) are nothing but the Dolbeault cohomology groups (resp. the (usual) holomorphic de Rham cohomology groups), and $H^q(M, E) \cong H^q(M, \Omega^0(E))$, where $\Omega^0(E)$ denotes the sheaf of germs of local holomorphic cross sections of E.

§2. Strongly pseudo-convex manifolds

2.1. Contact manifolds. Let M be a manifold and P a subbundle of $T(M)$. Put

$$P' = T(M) / P$$

and denote by $\tilde{\omega}$ the projection of $T(M)$ onto P'. For any $X, Y \in \Gamma(P)$, put

$$\omega(X, Y) = \tilde{\omega}([X, Y]).$$

Then we have $\omega(X, Y) = - \omega(Y, X)$ and $\omega(fX, Y) = f\omega(X, Y)$ $(f \in \mathcal{F}(M))$. Hence ω gives a cross section of $P' \otimes \Lambda^2 P^*$.

The subbundle P is called a contact structure if $\dim P' = 1$ and if ω_x is non-degenerate at each $x \in M$, i.e., the condition "$X \in P_x$ and $\omega(X, Y) = 0$ for all $Y \in P_x$" implies $X = 0$. And the manifold M together with the contact structure P is called a contact manifold. Note that a contact manifold is necessarily odd dimensional.

Let M be a contact manifold of dimension $2n-1$. A vector field X is called an infinitesimal contact transformation if it leaves the contact structure P invariant or $[X, \Gamma(P)] \subset \Gamma(P)$.

In what follows we assume that P' and hence $(P')^*$ are trivial. Let θ be a trivialization of $(P')^*$, by which we mean a cross section θ of $(P')^*$ such that $\theta_x \neq 0$ at each $x \in M$. Since $(P')^* \subset T(M)^*$ in a natural manner, θ is a 1-form on M. Since $d\theta(X, Y) = -\theta([X, Y])$ for all $X, Y \in \Gamma(P)$, we see that the restriction of $d\theta_x$ to P_x is non-degenerate at each $x \in M$. This clearly means that the $(2n-1)$-form $\theta \wedge (d\theta)^{n-1}$ gives a volume element on M or in other words, θ is a contact form. As is well known, it follows that there is a unique infinitesimal contact transformation ξ such that

$$\theta(\xi) = 1 \quad \text{and} \quad \xi \,\lrcorner\, d\theta = 0.$$

We notice that the assignment $\theta \longrightarrow \xi$ gives a one-to-one correspondence between the set of all trivializations θ of $(P')^*$ and the set of all infinitesimal contact transformations ξ such that $\xi_x \notin P_x$ at each $x \in M$.

2.2. Strongly pseudo-convex manifolds. Let M be a partially complex manifold. Let S be its partially complex structure and (P, I) its real expression. By (PC. 2) we have $[IX, IY] - [X, Y] \in \Gamma(P)$ for all $X, Y \in \Gamma(P)$, meaning that

$$\omega(IX, IY) = \omega(X, Y), \quad X, Y \in P_x.$$

For each $x \in M$, we define a P'_x-valued hermitian form L_x on P_x by

$$L_x(X, Y) = \omega(IX, Y).$$

The hermitian form L_x is usually called the Levi form at x. Especially if P is a contact structure and if P' is trivial, we have

$$L_x(X, Y) = - d\theta(IX, Y)\cdot\tilde{\omega}(\xi_x),$$

θ and ξ being as before.

We say that S is a strongly pseudo-convex (s. p. c.) structure and M is a s.p.c. manifold if $\dim P' = 1$ and if the Levi form L_x is definite at each x, i.e., the condition " $X \in P_x$ and $L_x(X, X) = 0$" implies $X = 0$. It should be noted that a s.p.c. manifold is a contact manifold, because P becomes a contact structure under this situation.

Let M be a s.p.c. manifold. Then we can easily prove the following

Proposition 2.1. P' is trivial and there is a trivialization θ of $(P')^*$ such that the hermitian quadratic form $-d\theta(IX, X)$, $X \in P_x$, is positive definite at each $x \in M$.

A trivialization θ of $(P')^*$ will be called a basic form if it satisfies the condition in Proposition 2.1. And an infinitesimal contact transformation ξ on the underlying contact manifold will be called a basic field if $\xi_x \notin P_x$ at each $x \in M$ and if the corresponding trivialization θ of $(P')^*$ is a basic form.

Let θ be a basic form. Then a 1-form θ' is a basic form if and only if $\theta' = f\theta$ with a positive function f. It follows that every s.p.c. manifold is oriented : θ being a basic form, the volume element $\theta \wedge (d\theta)^{n-1}$ gives the orientation.

2.3. S. p. c. real hypersurfaces. Let M' be an n-dimensional complex manifold, and S' its complex structure. Let f be a real valued function defined on an open set U of M' such that $df_x \neq 0$ at each $x \in U$. For each $x \in U$, define a subspace $S(f)_x$ of S'_x by

$$S(f)_x = \{ X \in S'_x \mid df(X) = 0 \},$$

and a hermitian form $L(f)_x$ on $S(f)_x$ by

$$L(f)_x(X, Y) = (d'd''f)(X, \bar{Y}), \qquad X, Y \in S(f)_x.$$

Assuming that $f^{-1}(0) \neq \phi$, let us consider the real hypersurface $M = f^{-1}(0)$. Let S be the partially complex structure of M, and (P, I) its real expression. Clearly we have $S_x = S(f)_x$, $x \in M$. Define a 1-form θ on M by

$$\theta = \sqrt{-1}\ \iota^* d''f = -\sqrt{-1}\ \iota^* d'f ,$$

where ι denotes the injection $M \longrightarrow M'$. Then θ is a real form and gives a trivialization of $(P')^*$. Furthermore we have

$$L(f)_x(X, Y) = -\sqrt{-1}\ d\theta(X, \bar{Y}), \qquad X, Y \in S_x.$$

Thus we know that M is s.p.c. as a partially complex manifold if and only if M is s.p.c. in M' in the classical sense.

Let M be a s.p.c. real hypersurface of a complex manifold M'. Then we shall say that an open set V of M' lies inside M if it satisfies the following conditions :

1) M is contained in the boundary ∂V of V ;

2) V is s.p.c. at each $x \in M$, i.e., there are a neighborhood U of x in M' and a real valued function f on U such that 1° : $df_x \neq 0$, 2° : $V \cap U = \{ y \in U \mid f(y) < 0 \}$, and 3° : $L(f)_x$ is positive definite (cf. Gunning-Rossi [4]).

§3. The canonical affine connections of strongly pseudo-convex manifolds.

3.1. The basic notations. Let M be a s.p.c. manifold of dimension $2n-1$, and ξ a basic field on M. Our discussions from now on will be concerned with the pair (M, ξ).

Let S be the s.p.c. structure of M and (P, I) its real expression. We denote by P' the 1-dimensional subbundle of $T(M)$ spanned by ξ : $P'_x = \mathbb{R}\xi_x$, $x \in M$. Clearly we have

$$T(M) = P + P' \quad \text{(direct sum)}.$$

θ denoting the basic form corresponding to ξ, we put

$$\omega = -d\theta .$$

Note that $\xi \lrcorner \omega = 0$. Let us now extend I to a tensor field of type $\binom{1}{1}$ in such a way that $I\xi = o$. Then we have

$$I^2X = -X + \theta(X)\xi , \qquad X \in T(M)_x$$

and

$$\omega(IX, IY) = \omega(X, Y), \qquad X,Y \in T(M)_x.$$

We define a tensor field g of type $\binom{0}{2}$ on M by

$$g(X, Y) = \omega(IX, Y).$$

Then g is symmetric and

$$g(IX, IY) = g(X, Y).$$

Furthermore, for each $x \in M$, the restriction of g_x to P_x is positive definite. (g is never a Riemannian metric, because $\xi \lrcorner g = 0$.)

3.2. The canonical affine connections.

We shall prove the following

Proposition 3.1. There is a unique affine connection

$$\nabla : \Gamma(T(M)) \longrightarrow \Gamma(T(M) \otimes T(M)^*)$$

on M satisfying the following conditions:

1) The contact structure P is parallel, i.e.,

$\nabla_X \Gamma(P) \subset \Gamma(P)$ for any $X \in \Gamma(T(M))$.

2) The tensor fields ξ, I and ω are all parallel, i.e., $\nabla\xi = \nabla I = \nabla\omega = 0$. (It follows also that $\nabla\theta = \nabla g = 0$.)

3) The torsion T of ∇ satisfies the equalities :

$$T(X, Y) = - \omega(X, Y)\xi ,$$

$$T(\xi, IY) = -IT(\xi, Y), \quad X, Y \in P_x.$$

Let $X, Y \in \Gamma(\mathbb{C}P)$. We denote by $[X, Y]_P$ (resp. by $[X, Y]_{P'}$ the $\mathbb{C}P$-component (resp. the $\mathbb{C}P'$-component) of $[X, Y]$ in the decomposition :

$$\mathbb{C}T(M) = \mathbb{C}P + \mathbb{C}P' \quad \text{(direct sum)}.$$

Clearly we have

$$[X, Y]_{P'} = \omega(X, Y)\xi .$$

We also denote by $[X, Y]_S$ (resp. by $[X, Y]_{\bar{S}}$) the S-component (resp. the $\bar{S}$-component) of $[X, Y]_P$ in the decomposition :

$$\mathbb{C}P = S + \bar{S} \quad \text{(direct sum)}.$$

Uniqueness. Let ∇ be any affine connection satisfying the conditions in Proposition 3.1. Let us extend ∇ to a differential operator of $\Gamma(\mathbb{C}T(M))$ to $\Gamma(\mathbb{C}T(M) \otimes (\mathbb{C}T(M))^*)$ in a natural manner. By 1) and $\nabla I = 0$, we have

$$\nabla_X\Gamma(S) \subset \Gamma(S),$$

$$\nabla_X\Gamma(\bar{S}) \subset \Gamma(\bar{S}), \quad X \in \Gamma(\mathbb{C}T(M)).$$

Lemma 3.2. Let $X, Y, Z \in \Gamma(S)$.

(1) $\nabla_{\bar{X}} Y = [\bar{X}, Y]_S$.

(2) $\omega(\nabla_X Y, \bar{Z}) = X\omega(Y, \bar{Z}) - \omega(Y, [X, \bar{Z}]_{\bar{S}})$.

(3) $\nabla_\xi Y = L_\xi Y + T_\xi Y$, where L_ξ denotes the Lie derivation and T_ξ is given by $T_\xi = -\frac{1}{2} I \cdot L_\xi I$.

Proof. Since $T(\bar{X}, Y) = -\omega(\bar{X}, Y)\xi$, we have

$$\nabla_{\bar{X}} Y - \nabla_Y \bar{X} = [\bar{X}, Y] + T(\bar{X}, Y) = [\bar{X}, Y]_P.$$

Hence $\nabla_{\bar{X}} Y = [\bar{X}, Y]_S$ and $\nabla_Y \bar{X} = [Y, \bar{X}]_{\bar{S}}$, proving (1). (2) is clear from the facts : $\nabla\omega = 0$ and $\nabla_X \bar{Z} = [X, \bar{Z}]_{\bar{S}}$. Since $\nabla\xi = 0$, we have

$$\nabla_\xi Y = [\xi, Y] + T(\xi, Y),$$

$$\nabla_\xi (IY) = [\xi, IY] + T(\xi, IY).$$

Since $T(\xi, IY) = -IT(\xi, Y)$ and $\nabla I = 0$, it follows that $(L_\xi I)Y - 2IT(\xi, Y) = 0$. Hence $T(\xi, Y) = T_\xi Y$, proving (3).

We have $\nabla\xi = 0$ and $\overline{\nabla_Z W} = \nabla_{\bar{Z}} \bar{W}$ for all $Z, W \in \Gamma(\mathbb{C}T(M))$. And the condition "$X \in S_x$ and $\omega(X, \bar{Y}) = 0$ for all $Y \in S_x$" implies $X = 0$. Therefore we see from Lemma 3.2 that ∇ is uniquely determined.

Existence. We first prepare two lemmas.

Lemma 3.3. (1) $IT_\xi = - T_\xi I$.

(2) If $Y \in \Gamma(S)$, then $L_\xi Y + T_\xi Y \in \Gamma(S)$.

(3) $L_\xi \omega = 0$.

(4) $\omega(T_\xi X, Y) + \omega(X, T_\xi Y) = 0, \qquad X, Y \in T(M)_x$.

Proof. We have $I^2 X = - X + \theta(X)\xi$ and $L_\xi \theta = 0$. Hence $(L_\xi I)I + I(L_\xi I) = 0$, which means (1). Let $Y \in \Gamma(S)$. By using (1), we see that $I(L_\xi Y + T_\xi Y) = L_\xi(IY) + T_\xi IY = \sqrt{-1}\,(L_\xi Y + T_\xi Y)$. Hence $L_\xi Y + T_\xi Y \in \Gamma(S)$, proving (2). We have $L_\xi \omega = -dL_\xi \theta = 0$, proving (3). Finally, (4) is easy from the facts : $L_\xi \omega = 0$ and $\omega(IX, IY) = \omega(X, Y)$.

Lemma 3.4. For any $X, Y, Z \in \Gamma(S)$, we have

$$X\,\omega(Y, \bar{Z}) + Y\,\omega(\bar{Z}, X) + \omega(\bar{Z}, [X, Y]) + \omega(X, [Y, \bar{Z}]_{\bar{S}}) + \omega(Y, [\bar{Z}, X]_{\bar{S}}) = 0.$$

This is easily obtained from the fact : $d\omega = - d^2\theta = 0$.

We are now in position to prove existence. We define a bilinear map $\nabla : \Gamma(\mathbb{C}T(M)) \times \Gamma(\mathbb{C}T(M)) \longrightarrow \Gamma(\mathbb{C}T(M))$ in the following manner : 1°. For any $X, Y \in \Gamma(S)$, $\nabla_{\bar{X}}Y$, $\nabla_X Y$ ($\in \Gamma(S)$) and $\nabla_\xi Y$ are respectively given by the equalities (1), (2) and (3) in Lemma 3.2 ; 2°. For any $Z \in \Gamma(\mathbb{C}T(M))$, $\nabla_Z \xi = 0$; 3°. For any $X, Y \in \Gamma(S)$, $\nabla_X \bar{Y} = \overline{\nabla_{\bar{X}} Y}$, $\nabla_{\bar{X}} \bar{Y} = \overline{\nabla_X Y}$ and $\nabla_\xi \bar{Y} = \overline{\nabla_\xi Y}$. Then it is easy to see that ∇ is the complexification of an affine connection and that it satisfies the following : $\nabla_X \omega = 0$, $T(\xi, X) = T_\xi X$ and $T(X, \bar{Y}) = -\omega(X, \bar{Y})$ for $X, Y \in S_x$. From Lemma 3.3, we get $\nabla_\xi \Gamma(S) \subset \Gamma(S)$ and $\nabla_\xi \omega = 0$, and from Lemma 3.4, $T(X, Y) = 0$ for $X, Y \in S_x$. Thus we see that ∇ satisfies the conditions 1), 2) and 3) in Proposition 3.1. We have thereby completed the proof of Proposition 3.1.

The affine connection ∇ in Proposition 3.1 will be called the canonical affine connection of (M, ξ).

Let ∇ be the canonical affine connection and R its curvature. Then we have $R(X, Y)P_x \subset P_x$ for all $X, Y \in T(M)_x$.

Proposition 3.5. Let $X_i \in P_x$ $(1 \leq i \leq 4)$.

(1) $R(X_1, X_2)IX_3 = IR(X_1, X_2)X_3$.

(2) $g(R(X_1, X_2)X_3, X_4) + g(X_3, R(X_1, X_2)X_4) = 0,$

(3) $\mathcal{S}\ R(X_1, X_2)X_3 = -\ \mathcal{S}\ \omega(X_1, X_2)T_\xi X_3,$ where $\mathcal{S}$ stands for the cyclic sum with respect to X_1, X_2 and X_3.

Corollary. Let $X_i \in S_x$ $(1 \leq i \leq 4)$.

(1) $R(X_1, \bar{X}_2)S_x \subset S_x$

(2) $R(X_1, \bar{X}_2)X_3 = R(X_3, \bar{X}_2)X_1.$

(3) $g(R(X_1, \bar{X}_2)X_3, \bar{X}_4) = g(R(X_3, \bar{X}_4)X_1, \bar{X}_2).$

The proof of these facts are left to the readers.

We now define an operator

$$R_* : \mathbb{C}P \longrightarrow \mathbb{C}P$$

as follows : Let $e_1, \ldots, e_{n-1}$ be any orthonormal base of S_x(with respect to g), i.e., a base of S_x such that $g(e_i, \bar{e}_j) = \delta_{ij}$. Then

$$R_* X = -\sqrt{-1}\sum_{i=1}^{n-1} R(e_i, \bar{e}_i)IX, \qquad X \in \mathbb{C}P_x.$$

Clearly R_* is a real operator, and by the corollary above we have

$$R_*\, S \subset S,$$

$$g(R_* X, \bar{Y}) = g(X, \overline{R_* Y}), \qquad X, Y \in S_x.$$

The operator R_* thus defined will be called the Ricci operator.

3.3. Green's theorem. We put

$$dv = \theta \wedge (d\theta)^{n-1},$$

which is a volume element on M. For $\alpha \in \Gamma(\bar{S}^*)$ we define a function $\delta''\alpha$ on M as follows : Let $x \in M$ and let $(e_1, \ldots, e_{n-1})$ be any orthonormal base of S_x. Then

$$(\delta''\alpha)(x) = - \sum_i (\nabla_{e_i} \alpha)(\bar{e}_i).$$

In the same way, for $\beta \in \Gamma(S^*)$, we define a function $\delta'\beta$ by

$$(\delta'\beta)(x) = - \sum_i (\nabla_{\bar{e}_i} \beta)(e_i).$$

Clearly we have $\delta'\beta = \overline{\delta''\bar{\beta}}$, where $\bar{\beta}$ is defined by $\bar{\beta}(\bar{X}) = \overline{\beta(X)}$ for all $X \in S_x$.

Proposition 3.6. Let $f \in \mathbb{C}F(M)$ and $\alpha \in \Gamma(\bar{S}^*)$.

(1) $\xi f \cdot dv = d(f \cdot \xi \lrcorner\, dv)$.

(2) $\delta''\alpha \cdot dv = - d\tilde{\alpha}$, where $\tilde{\alpha}$ is the $2(n-1)$-form defined by $\tilde{\alpha} = \sum_i \alpha(\bar{e}_i)\, e_i \lrcorner\, dv$.

Therefore if M is compact, we have

$$\int_M \xi f \cdot dv = \int_M \delta''\alpha \cdot dv = \int_M \delta'\beta \cdot dv = 0.$$

Proof. Since $L_\xi dv = 0$, we have $L_\xi(f\ dv) = \xi f \cdot dv$. We have $L_\xi(f\ dv) = d(f\xi \lrcorner dv)$, proving (1). We have $\nabla\theta = \nabla d\theta = 0$, whence $\nabla dv = 0$. Therefore

$$L_X dv = -\mathrm{Tr}A_X \cdot dv$$

for all $X \in \Gamma(\mathbb{C}T(M))$ ([10], Appendix 6), where A_X is the (complexified) tensor field of type $\binom{1}{1}$ defined by

$$A_X Y = -\nabla_Y X - T(X, Y), \quad Y \in \mathbb{C}T(M)_x.$$

Now define $X \in \Gamma(S)$ by $\alpha(\bar{Y}) = g(X, \bar{Y})$ for all $Y \in S_x$. Then $T(X, Y) \in \mathbb{C}P'_x$ if $Y \in \mathbb{C}P_x$, and $T(X, \xi) \in \mathbb{C}P_x$. Hence

$$\mathrm{Tr}A_X = -\sum_i g(\nabla_{e_i} X, \bar{e}_i) = \delta''\alpha.$$

Therefore $\delta''\alpha \cdot dv = -L_X dv = -d(X \lrcorner dv) = -d\tilde{\alpha}$, proving (2).

§4. The canonical connections of holomorphic vector bundles

4.1. Connections in complex vector bundles.

Let E be a complex vector bundle over a manifold M. A differential operator

$$D : \Gamma(E) \longrightarrow \Gamma(E \otimes \mathbb{C}T(M)^*)$$

is said to be a connection in E if it satisfies the following condition :

$$D(fu) = f\,Du + u \otimes df$$

for all $u \in \Gamma(E)$ and $f \in \mathbb{C}\mathcal{F}(M)$. As usual the covariant derivative $(Du)(X)$ of u in the direction $X \in \mathbb{C}T(M)_x$ will be denoted by $D_X u$. For any $X, Y \in \Gamma(\mathbb{C}T(M))$ and $u \in \Gamma(E)$, we put

$$K(X, Y)u = D_X(D_Y u) - D_Y(D_X u) - D_{[X, Y]}u.$$

Then K gives a cross section of $E \otimes E^* \otimes \Lambda^2(\mathbb{C}T(M))^*$, which is called the curvature of D.

Let us now fix an affine connection ∇ on M. As is well known, a connection D in E together with the affine connection ∇ gives rise to the covariant differentiation :

$$D : \Gamma(E^r_s \otimes (\mathbb{C}T(M))^k_\ell) \longrightarrow \Gamma(E^r_s \otimes (\mathbb{C}T(M))^k_{\ell+1}).$$

For example, let $\varphi \in \Gamma(E \otimes (\mathbb{C}T(M))^0_\ell)$. Then $D\varphi$ is defined by

$$(D\varphi)(X, X_1, \dots, X_\ell) = D_X(\varphi(X_1, \dots, X_\ell)) - \sum_i \varphi(X_1, \dots, \nabla_X X_i, \dots, X_\ell)$$

for all $X, X_1, \dots, X_\ell \in \Gamma(\mathbb{C}T(M))$. The covariant derivatives $(D\varphi)(X, \dots)$, $(D^2\varphi)(X, Y, \dots)$, etc. will be also written $(D_X\varphi)(\dots)$, $(D_X D_Y \varphi)(\dots)$, etc.

We shall frequently use the following

Lemma 4.1. For any $\varphi \in \Gamma(E \otimes T(M))^0_\ell)$ we have the equality (the Ricci formula) :

$$\begin{aligned}(D_X D_Y \varphi)(X_1, \dots, X_\ell) &= (D_Y D_X \varphi)(X_1, \dots, X_\ell) \\ &\quad - (D_{T(X, Y)}\varphi)(X_1, \dots, X_\ell) \\ &\quad + K(X, Y)\varphi(X_1, \dots, X_\ell) \\ &\quad - \sum_i \varphi(X_1, \dots, R(X, Y)X_i, \dots, X_\ell).\end{aligned}$$

4.2. The canonical connections. Let M be a s.p.c. manifold and ξ a basic field on M.

Proposition 4.2. Let E be a holomorphic vector bundle over the s.p.c. manifold M, and $< , >$ a hermitian inner product in E. Then there is a unique connection D in E satisfying the following conditions:

1) $D_{\bar{X}}u = \bar{X}\, u \; (= (\bar{\partial}_E u)(X)), \qquad u \in \Gamma(E), \quad X \in \Gamma(S);$

2) $X <u, u' > = < D_X u, u > + < u, D_{\bar{X}} u' >,$

$$u, u' \in \Gamma(E), \quad X \in \Gamma(\mathbb{C}T(M));$$

3) Let $x \in M$ and let $(e_1, \ldots, e_{n-1})$ be any orthonormal base of S_x. Then

$$\sum_i K(e_i, \bar{e}_i) = 0,$$

K being the curvature of D.

Proof. We first prove uniqueness. Let D be any connection in E satisfying the conditions in Proposition 4.2. By 1) and 2) we have

(4. 1) $\quad X <u, u'> = <D_X u, u' > + < u, \bar{X}u' >, \qquad X \in \Gamma(S).$

By Lemma 4.1, we have

$$K(X, Y)u = (D^2u)(X, Y) - (D^2u)(Y, X) + (Du)(T(X, Y)),$$

$$u \in \Gamma(E), \quad X, Y \in \Gamma(\mathbb{C}T(M)),$$

where the covariant differentiation D should be considered with respect to the canonical affine connection ∇ of (M, ξ). If we put $B(X, Y)u = (D^2u)(X, Y) - (D^2u)(Y, X)$, we have

$$(4.\ 2) \quad B(X, Y)u = (D_X(D_Y u) - D_{\nabla_X Y}u) - (D_Y(D_X u) - D_{\nabla_Y X}u).$$

Moreover since

$$\sum_i T(e_i, \bar{e}_i) = - \sum_i \omega(e_i, \bar{e}_i)\xi = (n - 1)\sqrt{-1}\,\xi,$$

it follows from 3) and the formula above for the curvature K that

$$(4.\ 3) \qquad D_\xi u = \frac{\sqrt{-1}}{n - 1} \sum_i B(e_i, \bar{e}_i)\, u.$$

Now we see from 1), (4. 1), (4. 2) and (4. 3) that D is uniquely determined. (Note that $\nabla_X Y \in \Gamma(\mathbb{C}P)$ if $X, Y \in \Gamma(\mathbb{C}P)$).

Let us now prove existence. We first define a bilinear map $D' : \Gamma(\mathbb{C}P) \times \Gamma(E) \ni (X, u) \longrightarrow D_X u \in \Gamma(E)$ in such a way that, for any $u \in \Gamma(E)$ and $X \in \Gamma(S)$, $D_{\bar{X}}u$ and $D_X u$ are respectively given by 1) and (4. 1). Using this map, we define $B(X, Y)u$ $(X, Y \in \Gamma(\mathbb{C}P))$ by (4. 2) and $D_\xi u$ by (4. 3). Now the map

D' together with the correspondence $u \longrightarrow D_\xi u$ gives rise to a bilinear map $D : \Gamma(\mathbb{C}T(M)) \times \Gamma(E) \longrightarrow \Gamma(E)$. It is easy to see that the bilinear map D, thus obtained, defines a connection in E and that it satisfies 1), 3) and the equality :

$$X < u, u' > = < D_X u, u' > + < u, \bar{X}u' > , \quad X \in \Gamma(\mathbb{C}P).$$

It follows from this equality that

$$(4.\ 4) \quad -T(X, Y) < u, u' > = < B(X, Y)u, u' > + < u, B(\bar{X}, \bar{Y})u' >,$$

$$X, Y \in \Gamma(\mathbb{C}P).$$

Therefore we see from (4. 3) that

$$\xi < u, u' > = < D_\xi u, u' > + < u, D_\xi u' >.$$

Thus D satisfies 2), completing the proof of Proposition 4.2.

The connection D in Proposition 4. 2 will be called the canonical connection of E with respect to the inner product $< , >$ and to the basic field ξ .

Proposition 4. 3. Let $X, Y \in S_x$ and $u, u' \in E_x$. The notations being as in Proposition 4. 2, then we have

(1) $K(X, Y) = K(\bar{X}, \bar{Y}) = 0.$

(2) $< K(X, \bar{Y})u, u' > + < u, K(\bar{X}, Y)u' > = 0.$

Proof. By 1) of Proposition 4. 2, we have $K(\bar{X}, \bar{Y}) = 0$, and by (4. 4),

$$< K(Z, W)u, u' > + < u, K(\bar{Z}, \bar{W})u'> = 0, \qquad Z, W \in \mathbb{C}P_x.$$

Hence we get (2) as well as $K(X, Y) = 0.$

II. The harmonic theory on strongly pseudo-convex manifolds

§5. The Laplacian

Let M be a s.p.c. manifold of dimension $2n - 1$, and ξ a basic field on M. For the pair (M, ξ) we use the same notations as in §4. Let E be a holomorphic vector bundle over M, and $\langle\, ,\, \rangle$ an inner product in E. D denotes the canonical connection of E with respect to $\langle\, ,\, \rangle$ and to ξ. The covariant differentiation D will be always considered with respect to the canonical affine connection ∇.

In this section we shall make a differential geometric study of the complex $\{ C^q(M, E), \bar\partial_E \}$ describing the Laplacian $\Box_E$ in terms of the covariant differentiation D.

5. 1. The fundamental operators. Since $\mathbb{C}T(M) = \mathbb{C}P' + S + \tilde{S}$ (direct sum), the vector bundle $C^q(M, E)$ may be identified with a subbundle of $E \otimes \Lambda^q(\mathbb{C}T(M))^*$, and, for any $X \in \Gamma(\mathbb{C}T(M))$ and $\varphi \in C^q(M, E)$, the covariant derivative $D_X\varphi$ is in $C^q(M, E)$.

For any $X, Y \in \Gamma(S)$, we have

$$\nabla_{\bar{X}}\bar{Y} - \nabla_{\bar{Y}}\bar{X} = [\bar{X}, \bar{Y}],$$

because $T(\bar{X}, \bar{Y}) = 0$. Hence the operator $\bar{\partial}_E : C^q(M, E) \longrightarrow C^{q+1}(M, E)$ may be expressed as follows :

$$(\bar{\partial}_E\varphi)(\bar{X}_1, \ldots, \bar{X}_{q+1}) = \sum_j (-1)^{j+1}(D_{\bar{X}_j}\varphi)(\bar{X}_1, \ldots, \hat{\bar{X}}_j, \ldots, \bar{X}_{q+1}),$$

where $\varphi \in C^q(M, E)$ and $X_1, \ldots, X_{q+1} \in S_x$.

The inner product $< , >$ together with the tensor field g induces an inner product $< , >$ in the vector bundle $C^q(M, E)$: Let $\varphi, \psi \in C^q(M, E)_x$ and let $(e_1, \ldots, e_{n-1})$ be any orthonormal base of S_x. Then

$$< \varphi, \psi > = \frac{1}{q!} \sum_{i_1, \ldots, i_q} < \varphi(\bar{e}_{i_1}, \ldots, \bar{e}_{i_q}), \psi(\bar{e}_{i_1}, \ldots, \bar{e}_{i_q}) > .$$

The operator ϑ_E. We define a differential operator

$$\vartheta_E : C^{q+1}(M, E) \longrightarrow C^q(M, E)$$

by

$$(\vartheta_E\varphi)(\bar{X}_1, \ldots, \bar{X}_q) = - \sum_i (D_{e_i}\varphi)(\bar{e}_i, \bar{X}_1, \ldots, \bar{X}_q),$$

where $\varphi \in C^q(M, E)$. For any $\varphi \in C^q(M, E)$ and $\psi \in C^{q+1}(M, E)$, we have

$$(5.\ 1) \qquad < \bar{\partial}_E\varphi, \psi> = < \varphi, \partial_E\psi > - \delta'\alpha ,$$

where α is given by

$$\alpha(X) = \langle \varphi, \bar{X} \lrcorner \psi \rangle, \quad X \in S_x.$$

The operator $\Box_E$. The differential operator

$$\vartheta_E \bar{\partial}_E + \bar{\partial}_E \vartheta_E : C^q(M, E) \longrightarrow C^q(M, E)$$

is called the Laplacian and is denoted by $\Box_E$. A solution φ of the equation $\Box_E \varphi = 0$ is called a harmonic form. We denote by $H^q(M, E)$ the space of harmonic forms in $C^q(M, E)$.

The Ricci operator R_*. Using the Ricci operator R_* (see 3.2), we define an operator

$$R_* : C^q(M, E) \longrightarrow C^q(M, E)$$

by

$$(R_*\varphi)(\bar{X}_1, \ldots, \bar{X}_q) = \sum_j \varphi(\bar{X}_1, \ldots, R_*\bar{X}_j, \ldots, \bar{X}_q)$$

for all $\varphi \in C^q(M, E)_x$ and $X_1, \ldots, X_q \in S_x$. Since $g(R_*X, \bar{Y}) = g(X, R_*\bar{Y})$ for all $X, Y \in S_x$, we see that the operator R_* is self-adjoint with respect to the inner product $\langle\ ,\ \rangle$ in $C^q(M, E)$.

The operator K. Using the curvature K of D, we define an operator

$$K : C^q(M, E) \longrightarrow C^q(M, E)$$

by

$$(K\varphi)(\bar{X}_1,..,\bar{X}_q) = \sum_{i,j} (-1)^{j+1} K(e_i, \bar{X}_j)\, \varphi(\bar{e}_i, \bar{X}_1, \ldots, \hat{\bar{X}}_j \ldots, \bar{X}_q)$$

By Proposition 4.3, we see that the operator K is self-adjoint

The operator Q^q. For any integer q, we define a self-adjoint operator

$$Q^q : C^q(M, E) \longrightarrow C^q(M, E)$$

by

$$Q^q = K + \frac{n-q-1}{n-1} R_* .$$

5.2. The description of $\Box_E$ in terms of D. The main aim of this paragraph is to prove the following

Proposition 5.1. For any $\varphi \in C^q(M, E)$, we have the equalities :

(1) $\qquad \Box_E \varphi = -\sum_i D_{e_i} D_{\bar{e}_i} \varphi - q\sqrt{-1}\, D_\xi \varphi + K\varphi + R_* \varphi .$

(2) $\qquad \Box_E \varphi = -\sum_i D_{\bar{e}_i} D_{e_i} \varphi + (n-q-1)\sqrt{-1}\, D_\xi \varphi + K\varphi .$

(3) $\qquad \Box_E \varphi = -\frac{n-q-1}{n-1} \sum_i D_{e_i} D_{\bar{e}_i} \varphi - \frac{q}{n-1} \sum_i D_{\bar{e}_i} D_{e_i} \varphi + Q^q \varphi .$

Proof. (3) is obtained from (1) and (2) by eliminating $\sqrt{-1}\, D_\xi \varphi$.

Let $X_1, \ldots, X_q \in S_x$. We have

(5.2) $\qquad (\vartheta_E \bar{\partial}_E \varphi)(\bar{X}_1, \ldots, \bar{X}_q)$

$$= -\sum_i (D_{e_i} \bar{\partial}_E \varphi)(\bar{e}_i, \bar{X}_1, \ldots, \bar{X}_q)$$

$$= -\sum_i (D_{e_i} D_{\bar{e}_i} \varphi)(\bar{X}_1, \ldots, \bar{X}_q)$$

$$- \sum_i \sum_j (-1)^j (D_{e_i} D_{\bar{X}_j} \varphi)(\bar{e}_i, \bar{X}_1, \ldots, \hat{\bar{X}}_j, \ldots, \bar{X}_q).$$

If we put

$$A_j = \sum_i (D_{e_i} D_{\bar{X}_j} \varphi)(\bar{e}_i, \bar{X}_1, \ldots, \hat{\bar{X}}_j, \ldots, \bar{X}_q) ,$$

it follows from the Ricci formula (Lemma 4.1) that

$$A_j = A_j^1 + A_j^2 + \dots + A_j^6, \quad \text{where}$$

$$A_j^1 = \sum_i (D_{\bar{X}_j} D_{e_i} \varphi)(\bar{e}_i, \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_q),$$

$$A_j^2 = \sum_i \omega(e_i, \bar{X}_j)(D_\xi \varphi)(\bar{e}_i, \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_q),$$

$$A_j^3 = \sum_i K(e_i, \bar{X}_j)\varphi(\bar{e}_i, \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_q),$$

$$A_j^4 = - \sum_i \varphi(R(e_i, \bar{X}_j)\bar{e}_i, \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_q),$$

$$A_j^5 = - \sum_{k<j} \sum_i \varphi(\bar{e}_i, \bar{X}_1, \dots, \underset{\substack{\uparrow \\ k^{th} \text{ place}}}{R(e_i, \bar{X}_j)\bar{X}_k}, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_q),$$

$$A_j^6 = - \sum_{k>j} \sum_i \varphi(\bar{e}_i, \bar{X}_1, \dots, \hat{\bar{X}}_j, \dots, \underset{\substack{\uparrow \\ k^{th} \text{ place}}}{R(e_i, \bar{X}_j)\bar{X}_k}, \dots, \bar{X}_q) .$$

First of all we have

$$\sum_j (-1)^j A_j^1 = (\bar{\partial}_E \vartheta_E \varphi)(\bar{X}_1, \dots, \bar{X}_q).$$

Since $\sum_i \omega(e_i, \bar{X}_j)\, \bar{e}_j = -\sqrt{-1}\, \bar{X}_j$, we obtain

$$\sum_j (-1)^j A_j^2 = q \sqrt{-1}\, (D_\xi \varphi)(\bar{X}_1, \dots, \bar{X}_q).$$

We have

$$\sum_j (-1)^j A_j^3 = - (K\varphi)(\bar{X}_1, \dots, \bar{X}_q).$$

By Corollary to Proposition 3.5, we have $\sum_i R(e_i, \bar{X}_j)\bar{e}_i$

$= - R_* \bar{X}_j$ and hence

$$\sum_j (-1)^j A_j^4 = - (R_* \varphi)(\bar{X}_1, \ldots, \bar{X}_q).$$

Furthermore by using the fact that $R(e_i, \bar{X}_k)\bar{X}_j = R(e_i, \bar{X}_j)\bar{X}_k$

(the corollary, i.b.i.d.), we can easily show

$$\sum_j (-1)^j A_j^5 + \sum_j (-1)^j A_j^6 = 0.$$

We have thus proved the equality:

$$\sum_j (-1)^j A_j = (\bar{\partial}_E \vartheta_E \varphi + q\sqrt{-1}\, D_\xi \varphi - K\varphi - R_*\varphi)(\bar{X}_1, \ldots, \bar{X}_q).$$

and (1) follows from this equality and (5.2).

By the Ricci formula we obtain

$$\begin{aligned} &\sum_i (D_{e_i} D_{\bar{e}_i} \varphi)(\bar{X}_1, \ldots, \bar{X}_q) \\ &= \sum_i (D_{\bar{e}_i} D_{e_i} \varphi)(\bar{X}_1, \ldots, \bar{X}_q) + \sum_i \omega(e_i, \bar{e}_i)(D_\xi \varphi)(\bar{X}_1, \ldots, \bar{X}_q) \; . \\ &+ \sum_i K(e_i, \bar{e}_i)\varphi(\bar{X}_1, \ldots, \bar{X}_q) - \sum_i \varphi(\bar{X}_1, \ldots, R(e_i, \bar{e}_i)\bar{X}_j, \ldots, \bar{X}_q) \end{aligned}$$

Since $\sum_i K(e_i, \bar{e}_i) = 0$ and $\sum_i \omega(e_i, \bar{e}_i) = -(n-1)\sqrt{-1}$, it follows

that

$$\sum_i (D_{e_i} D_{\bar{e}_i} \varphi) = \sum_i D_{\bar{e}_i} D_{e_i} \varphi - (n-1)\sqrt{-1}\, D_\xi \varphi + R_* \varphi \, .$$

(2) now follows from this equality and (1). We have thereby

completed the proof of Proposition 5.1.

5.3. The fundamental equalities. From now on we assume that M is compact. We define an inner product $(\ ,\ )$ in the space $C^q(M, E)$ by

$$(\varphi, \psi) = \int_M \langle\varphi, \psi\rangle \ dv$$

for all $\varphi, \psi \in C^q(M, E)$. By (5.1) and Proposition 3.6, we see that ϑ_E is the (formal) adjoint operator of $\bar\partial_E$.

Let us now define semi-norms $\|\ \|_{\bar S}$ and $\|\ \|_S$ in $C^q(M, E)$ as follows :

$$\|\varphi\|_{\bar S}^2 = \int_M \left(\sum_i \langle D_{\bar e_i}\varphi, D_{\bar e_i}\varphi\rangle \right) dv ,$$

$$\|\varphi\|_S^2 = \int_M \left(\sum_i \langle D_{e_i}\varphi, D_{e_i}\varphi\rangle \right) dv .$$

We have

$$\sum_i \langle D_{\bar e_i}\varphi, D_{\bar e_i}\varphi\rangle = - \sum_i \langle D_{e_i} D_{\bar e_i}\varphi, \varphi\rangle - \delta''\alpha ,$$

where α is given by

$$\alpha(\bar X) = \langle D_{\bar X}\varphi, \varphi\rangle , \qquad X \in S_x .$$

Accordingly from Proposition 3.6, we get

$$\| \varphi \|_{\bar{S}}^{2} = - \int (\sum_i < D_{e_i} D_{\bar{e}_i} \varphi , \varphi >) \, dv.$$

In the same way,

$$\| \varphi \|_{S}^{2} = - \int (\sum_i < D_{\bar{e}_i} D_{e_i} \varphi , \varphi >) \, dv.$$

Therefore by Proposition 5.1, we have the following

Theorem 5.2. For any $\varphi \in C^q(M, E)$ we have the equalities :

(1) $(\square_E \varphi , \varphi) = \| \varphi \|_{\bar{S}}^{2} - q(\sqrt{-1}\, D_\xi \varphi, \varphi) + (K \varphi + R_* \varphi, \varphi).$

(2) $(\square_E \varphi , \varphi) = \| \varphi \|_{S}^{2} + (n-q-1)(\sqrt{-1}\, D_\xi \varphi, \varphi) + (K\varphi, \varphi).$

(3) $(\square_E \varphi , \varphi) = \dfrac{n-q-1}{n-1} \| \varphi \|_{\bar{S}}^{2} + \dfrac{q}{n-1} \| \varphi \|_{S}^{2} + (Q^q \varphi, \varphi).$

As an immediate consequence of Theorem 5.2, we get

Proposition 5.3. Assume that, for some q, the self-adjoint operator Q_x^q is positive definite at each $x \in M$. Then we have $H^q(M, E) = 0$.

§6. The harmonic theory for the complex $\{C^q(M, E), \bar\partial_E\}$

We preserve the notations in the previous section. For simplicity we put $m = 2n - 1$.

6.1. The Sobolev norms (Hörmander [7]). As usual, let $C_0^\infty(\mathbb{R}^m)$ denote the space of C^∞ functions with compact support on $\mathbb{R}^m$. For each real number s, we define the Sobolev norm $\| \ \|_{(s)}$ in $C_0^\infty(\mathbb{R}^m)$ by

$$\| f \|_{(s)}^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi , \quad f \in C_0^\infty(\mathbb{R}^m),$$

where $\hat{f}(\xi)$ is the Fourier transform of f, i.e.,

$$\hat{f}(\xi) = (2\pi)^{-\frac{m}{2}} \int e^{-\sqrt{-1} \langle x, \xi \rangle} f(x) \, dx.$$

Let $\{U_k, h_k\}_{k \in K}$ be an atlas of M such that K is a finite set and such that each U_k is homeomorphic with $\mathbb{R}^m$. (We are assuming that M is compact.) Let $\{\rho_k\}$ be a partition of unity subordinate to the atlas. For each k, take a moving frame $(e_1^k, \ldots, e_{n-1}^k)$ of $S|U_k$ and a moving frame $(s_1^k, \ldots, s_r^k)$ of $E|U_k$ $(r = \dim_{\mathbb{C}} E)$.

Now denote by $\mathcal{J}^q$ the family of all ordered set $(i_1, \ldots, i_q)$ of integers with $1 \leq i_1 < \ldots < i_q \leq n-1$. For any $\varphi \in C^q(M, E)$, $I \in \mathcal{J}^q$ and ℓ $(1 \leq \ell \leq r)$, define a function $\varphi^k_{\ell, I}$ on U_k by

$$\varphi(\bar{e}_{i_1}, \ldots, \bar{e}_{i_q}) = \sum_{\ell} \varphi^k_{\ell, I} \, s^k_{\ell},$$

where $I = (i_1, \ldots, i_q)$. By using the functions $(\rho_k \cdot \varphi^k_{\ell, I}) \circ h_k^{-1}$ in $C_0^\infty(\mathbb{R}^m)$, we now define the Sobolev norm $\| \ \|_{(s)}$ in $C^q(M, E)$ by

$$\| \varphi \|^2_{(s)} = \sum_{\ell, I, k} \| (\rho_k \cdot \varphi^k_{\ell, I}) \circ h_k^{-1} \|^2_{(s)}.$$

6.2. Kohn's harmonic theory. Hereafter we fix an integer q with $1 \leq q \leq n-2$. First of all we define norms $\| \ \|$ and $||| \ |||$ in $C^q(M, E)$ respectively by

$$\| \varphi \|^2 = (\varphi, \varphi),$$

$$||| \varphi |||^2 = \| \varphi \|^2_{\bar{S}} + \| \varphi \|^2_{S} + \| \varphi \|^2.$$

Note that the norms $\| \ \|$ and $\| \ \|_{(0)}$ are equivalent.

Proposition 6.1.

$$||| \varphi |||^2 \leq C(\Box_E \varphi, \varphi) + \| \varphi \|^2), \qquad \varphi \in C^q(M, E).$$

This follows immediately from Theorem 5.2, because

$1 \le q \le n-2$ and the operator Q^q is of order 0.

Proposition 6.2.

$$\| \varphi \|_{(\frac{1}{2})} \le C ||| \varphi |||, \quad \varphi \in C^q(M, E).$$

Proof. k being fixed, let K be any compact subset of U_k. Put $C_0^q(K) = \{\varphi \in C^q(M, E) \mid \operatorname{supp} \varphi \subset K\}$. Now the system $(X_1, \ldots, X_{m-1}) = (e_1^k, \ldots, e_{n-1}^k, \overline{e_1^k}, \ldots, \overline{e_{n-1}^k})$ gives a moving frame of $\mathbb{C}P|U_k$. Since P is a contact structure, we know that, at each $x \in U_k$, the complexified tangent space $\mathbb{C}T(M)_x$ is spanned by the vectors of the form : $(X_i)_x$, $[X_j, X_k]_x$, $1 \le i,j,k \le m-1$. Therefore by Kohn's inequality (Folland-Kohn [2], Theorem 5.4.7), we have

$$\| \varphi^k_{\ell,I} \|^2_{(\frac{1}{2})} \le C_1 \Big(\sum_{i=1}^{m-1} \| X_i \varphi^k_{\ell,I} \|^2 + \| \varphi^k_{\ell,I} \|^2 \Big), \quad \varphi \in C_0^q(K).$$

Clearly we have

$$\sum_{i,\ell,I} \| (D_{X_i} \varphi)^k_{\ell,I} \|^2 + \sum_{\ell,I} \| \varphi^k_{\ell,I} \|^2 \le C_2 ||| \varphi |||^2 , \quad \varphi \in C_0^q((K),$$

and the functions $(D_{X_i} \varphi)^k_{\ell,I}$ may be expressed as

$$(D_{X_i} \varphi)^k_{\ell,I} = X_i \varphi^k_{\ell,I} + \sum_{\ell',I'} a^{\ell',I'}_{i,\ell,I} \varphi^k_{\ell' I'} , \qquad \text{where } a^{\ell',I'}_{i,\ell,I}$$

are functions on U_k. From these it follows that $\| \varphi \|_{(\frac{1}{2})} \le C_3 ||| \varphi |||$,

$\varphi \in C_0^q(K)$. Thus we get Proposition 6.2.

By definition, the operator $\square_E$ is *subelliptic* if we can find a positive number σ such that

$$\|\varphi\|_{(\sigma)}^2 \leq C_\sigma ((\square_E\varphi, \varphi) + \|\varphi\|^2), \qquad \varphi \in C^q(M, E).$$

Theorem 6.3 (cf. Kohn [13] and Folland-Kohn [2]). The operator $\square_E$ is subelliptic with $\sigma = \frac{1}{2}$.

This follows immediately from Propostions 6.1 and 6.2. By virtue of Theorem 6.3, we have the following important result :

Theorem 6.4 (Kohn-Nirenberg [14], Kohn [13], Folland-Kohn [2], and Hörmander [9]). The operator $\square_E$ is hypoelliptic.

By Theorems 6.3 and 6.4 and by using standard arguments, we now arrive at the main theorem in the harmonic theory, essentially due to Kohn [13].

Theorem 6.5. (1) $H^q(M, E)$ is finite dimensional for any q with $1 \leq q \leq n-2$.

(2) For each q with $1 \le q \le n-2$, there are unique operators

$$H,\ G : C^q(M, E) \longrightarrow C^q(M, E)$$

such that

$$\square_E H = HG = 0, \qquad \square_E G + H = 1.$$

The operator G is usually called the Green operator.

Assuming that $n \ge 3$, let us consider the case where $q = 0$ or $n-1$. We first define an operator

$$H : C^0(M, E) \longrightarrow C^0(M, E)$$

by

$$H\varphi = \varphi - \vartheta G \bar{\partial} \varphi, \qquad \varphi \in C^0(M, E).$$

Then we easily find that H is an orthogonal projection of $C^0(M, E)$ onto $\mathcal{H}^0(M, E)$. In the same way we define an operator

$$H : C^{n-1}(M, E) \longrightarrow C^{n-1}(M, E)$$

by

$$H\varphi = \varphi - \bar{\partial} G \vartheta \varphi, \qquad \varphi \in C^{n-1}(M, E),$$

which is an orthogonal projection of $C^{n-1}(M, E)$ onto $\mathcal{H}^{n-1}(M, E)$.

Theorem 6.5 combined with the remark above gives the following

Proposition 6.6 (cf. Kohn [13]). Assume that $n \geq 3$, and let q be any integer. Then every cohomology class in $H^q(M, E)$ is represented by a unique harmonic form. Hence

$$\mathcal{H}^q(M, E) \cong H^q(M, E).$$

Remark. In the proof of Theorem 6.3, Kohn's inequality played an important role. We can also arrive at the subellipticity by utilizing Hörmander's inequality (Appendix, Theorem 9). In Appendix we shall give a simple and geometric proof of Hörmander's inequality and argue about some related problems.

§7. The cohomology groups $H^{p,q}(M)$.

Let M be a compact s.p.c. manifold of dimension $m = 2n - 1$, and ξ a basic field on M.

7.1. The complex $\{C^{p,q}(M), d''\}$. We define a Riemannian metric h on M by $h = g + \theta^2$ or in other words,

$$h(\xi, \xi) = 1, \quad h(\xi, X) = 0,$$

$$h(X, Y) = g(X, Y), \qquad X, Y \in P_x.$$

The Riemannian metric h induces a hermitian inner product $< , >$ in the complexified tangent bundle $\mathbb{C}T(M)$:

$$<X, Y> = h(X, \bar{Y}), \qquad X, Y \in \mathbb{C}T(M)_x.$$

And this inner product in turn induces an inner product $< , >$ in the vector bundle $A^k(M) = \Lambda^k(\mathbb{C}T(M))^*$: Let φ, $\psi \in A^k(M)_x$ and let $(e_1, \ldots, e_m)$ be an orthonormal base of $\mathbb{C}T(M)_x$. Then

$$<\varphi, \psi> = \frac{1}{k!} \sum_{i_1,\ldots,i_k} \varphi(e_{i_1}, \ldots, e_{i_k}) \overline{\psi(e_{i_1}, \ldots, e_{i_k})} .$$

Furthermore h together with the orientation of M gives the $*$ - operator

$$*_A \;:\; A^k(M) \longrightarrow A^{m-k}(M).$$

We have

$$dv = (n-1)!\; *_A 1.$$

These being said, we now define an inner product $(\;,\;)$ in the space $A^k(M)$ by

$$(\varphi,\psi) = \int <\varphi,\psi>dv = (n-1)!\int \varphi \wedge \overline{*_A\psi},$$

for all $\varphi,\psi \in A^k(M)$.

We have

$$\mathbb{C}T(M) = \mathbb{C}P' \oplus S \oplus \bar{S},$$

where the symbol $\oplus$ means that the sum is orthogonal with respect to the inner product $<\;,\;>$. Accordingly we may identify $\hat{T}(M) = \mathbb{C}T(M)/\bar{S}$ with the subbundle $\mathbb{C}P' \oplus S$ of $\mathbb{C}T(M)$, and hence $C^{p,q}(M) = \Lambda^p\hat{T}(M)^* \otimes \Lambda^q\bar{S}^*(M)$. More precisely $C^{p,q}(M)$ is a subbundle of $A^{p,q}(M)$ and we have the decomposition :

$$A^{p,q}(M) = A^{p+1,q-1}(M) \oplus C^{p,q}(M).$$

Let ι^p be the projection of $A^{p,q}(M)$ onto $C^{p,q}(M)$, which just corresponds to the natural map $A^{p,q}(M) \longrightarrow C^{p,q}(M)$ given in the proof of Proposition 1.1. Then we have

$$d''\varphi = \iota^p d\varphi, \qquad \varphi \in C^{p,q}(M).$$

If $\varphi \in C^{p,q}(M)$ and $\psi \in C^{r,s}(M)$, then $\varphi \wedge \psi \in C^{p+r,q+s}(M)$ and

$$d''(\varphi \wedge \psi) = d''\varphi \wedge \psi + (-1)^{p+q}\varphi \wedge d''\psi .$$

Now consider the anti-isomorphism

$$\#_A : A^k(M) \ni \varphi \longrightarrow \overline{*_A\varphi} \in A^{m-k}(M).$$

Lemma 7.1. For any (p, q), we have

$$\#_A C^{p,q}(M) = C^{n-p,\, n-q-1}(M).$$

Proof. We have

$$C^{p,q}(M) = \oplus \sum_{r+s=p} B(r,s,q),$$

where

$$B(r,s,q) = \Lambda^r(\mathbb{C}P')^* \otimes \Lambda^s S^* \otimes \Lambda^q \bar{S}^*.$$

It is easy to see that $\#_A B(r,s,q) = B(1-r,\, n-s-1,\, n-q-1)$. Hence

$$\#_A C^{p,q}(M) = \oplus \sum_{r+s=p} B(1-r,\, n-s-1,\, n-q-1) = C^{n-p,n-q-1}(M).$$

Lemma 7.2. Let δ'' be the formal adjoint operator of $d'' : C^{p,q}(M) \longrightarrow C^{p,q+1}(M)$ with respect to the inner product $(\ ,\ )$. Then we have

$$\delta''\psi = (-1)^{p+q+1} \#_A d''\#_A\psi , \qquad \psi \in C^{p,q+1}(M).$$

Proof. Let $\varphi \in C^{p,q}(M)$ and $\psi \in C^{p,q+1}(M)$. Then

$$d(\varphi \wedge \#_A\psi) = d''(\varphi \wedge \#_A\psi)$$

$$= d''\varphi \wedge \#_A\psi + (-1)^{p+q}\varphi \wedge d''\#_A\psi$$

and $d''\#_A\psi = \#_A \#_A d'' \#_A\psi$. Therefore it follows from the Stokes theorem that $(d''\varphi, \psi) = (\varphi, (-1)^{p+q+1} \#_A d''\#_A \psi)$, proving Lemma 7.2.

7.2. The harmonic theory and the duality. Let us now consider the complex $\{C^q(M, E^p), \bar{\partial}_{E^p}\}$ and apply the results in §§5 and 6, where the holomorphic vector bundle $E^p = \Lambda^p \hat{T}(M)^*$ should be equipped with the inner product $\langle\ ,\ \rangle$ as a subbundle of $A^p(M)$. By Proposition 1.1 we have $C^{p,q}(M) = C^q(M, E^p)$ and $d'' = (-1)^p\bar{\partial}_{E^p}$. It is clear that the inner product in $C^q(M, E^p)$ induced from the inner product in E^p just coincides with the inner product $\langle\ ,\ \rangle$ in $C^{p,q}(M)$. Hence

$$\delta'' = (-1)^p\vartheta_{E^p} .$$

We put as follows :

$$\Delta'' = \delta'' d'' + d'' \delta''.$$

$$\mathcal{H}^{p,q}(M) = \{ \varphi \in C^{p,q}(M) \mid \Delta'' \varphi = 0 \} .$$

Then $\Delta'' = \square_{E^p}$ and $\mathcal{H}^{p,q}(M) = \mathcal{H}^q(M, E^p)$. Therefore by the general theory developed in §6, we have $\dim \mathcal{H}^{p,q}(M) < \infty$ if $q \neq 0$, $n-1$, and $H^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$.

By Lemmas 7.1 and 7.2, we have the following

Theorem 7.3 (cf. the Serre duality). For any (p, q) we have

$$\#_A \mathcal{H}^{p,q}(M) = \mathcal{H}^{n-p,n-q-1}(M).$$

Hence

$$H^{p,q}(M) \cong H^{n-p,n-q-1}(M).$$

Finally we add the following

Proposition 7.4. Assume that the Ricci operator R_* is positive definite, i.e., the quadratic form $g(R_* X, \bar{X})$, $X \in S_x$, is positive definite at each $x \in M$. Then we have $H^{0,q}(M) = 0$ for any q with $q \neq 0$, $n-1$.

Proof. Apply Proposition 5.3 to the trivial holomorphic vector bundle $M \times \mathbb{C}$ (with the usual inner product).

For example, consider the unit sphere S^{2n-1} in $\mathbb{C}^n$, which is a compact s.p.c. real hypersurface. Let ξ be the vector field on S^{2n-1} induced from the 1-parameter group of transformations $\mathbb{R} \times S^{2n-1} \ni (t, x) \to e^{t\sqrt{-1}} \cdot x \in S^{2n-1}$. Then it can be shown that ξ is a basic field and that the associated Ricci operator R_* is positive definite. Hence $H^{0,q}(S^{2n-1}) = 0$ for any q with $q \neq 0$, $n-1$ by Proposition 7.4. Moreover since $\hat{T}(S^{2n-1})$ $(= \hat{T}(\mathbb{C}^n)|S^{2n-1})$ is holomorphically trivial, we have $H^{p,q}(S^{2n-1}) = 0$ for any (p, q) with $q \neq 0$, $n-1$ (cf. Theorem 10.3).

§8. The cohomology groups $H_*^{k-1,1}(M)$ and $H_0^k(M)$

We use the same notations as in the previous section.

8.1. The complex $\{ A^{p,q}(M), d \}$. Since $A^{p,q}(M) = A^{p+1,q-1}(M) \oplus C^{p,q}(M)$, we have the decomposition :

$$A^{p,q}(M) = \oplus \sum_{i=0}^{q} C^{p+q-i,i}(M).$$

Let $\varphi \in C^{p,q}(M)$. Then $d\varphi \in A^{p,q+1}(M)$ or more precisely

$$d\varphi \in C^{p+2,q-1}(M) \oplus C^{p+1,q}(M) \oplus C^{p,q+1}(M).$$

Consequently $d\varphi$ can be written uniquely in the form :

$$d\varphi = A\varphi + d'\varphi + d''\varphi ,$$

where $A\varphi \in C^{p+2,q-1}(M)$ and $d'\varphi \in C^{p+1,q}(M)$.

In general let $\varphi \in A^k(M)$ and $X_1, \ldots, X_k \in \mathbb{C}T(M)_x$. Then we have

$$(d\varphi)(X_1, \ldots, X_{k+1}) = \sum_{\lambda} (-1)^{\lambda+1} (\nabla_{X_\lambda} \varphi)(X_1, \ldots, \hat{X}_\lambda, \ldots, X_{k+1})$$
$$+ \sum_{\lambda<\mu} (-1)^{\lambda+\mu+1} \varphi(T(X_\lambda, X_\mu), X_1, \ldots, \hat{X}_\lambda, \ldots, \hat{X}_\mu, \ldots, X_{k+1}).$$

Now we know the following (see Proposition 3.1) : 1°. $T(X, Y) \in \bar{S}_x$ if $X, Y \in \hat{T}(M)_x$; 2°. $T(X, \bar{Y}) \in \hat{T}(M)_x$ if $X \in \hat{T}(M)_x$ and

$Y \in S_x$; 3°. $T(\bar{X}, \bar{Y}) = 0$ if $X, Y \in S_x$. Therefore, for any $\varphi \in C^{p,q}(M)$, $A\varphi$, $d'\varphi$ and $d''\varphi$ may be described respectively as follows : (In the following, $X_1, X_2, \dots$ (resp. $Y_1, Y_2, \dots$) denote any vectors in $\hat{T}(M)_x$ (resp. in S_x) at any $x \in M$.)

$$(8.1) \qquad (A\varphi)(X_1, \dots, X_{p+2}, \bar{Y}_1, \dots, \bar{Y}_{q-1})$$
$$= \sum_{i<j} (-1)^{i+j+1} \varphi(T(X_i, X_j), X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+2}, \bar{Y}_1, \dots, \bar{Y}_{q-1})$$

$$(8.2) \qquad (d'\varphi)(X_1, \dots, X_{p+1}, \bar{Y}_1, \dots, \bar{Y}_q)$$
$$= \sum_i (-1)^{i+1} (\nabla_{X_i}\varphi)(X_1, \dots, \hat{X}_i, \dots, X_{p+1}, \bar{Y}_1, \dots, \bar{Y}_q).$$

$$(8.3) \qquad (d''\varphi)(X_1, \dots, X_p, \bar{Y}_1, \dots, \bar{Y}_{q+1})$$
$$= (-1)^p \sum_i (-1)^{j+1} (\nabla_{\bar{Y}_j}\varphi)(X_1, \dots, X_p, \bar{Y}_1, \dots, \hat{\bar{Y}}_j, \dots, \bar{Y}_{q+1})$$
$$+ (-1)^p \sum_{i,j} (-1)^{i+j+1} \varphi(T(X_i, \bar{Y}_j), X_1, \dots, \hat{X}_i, \dots, X_p, \bar{Y}_1, \dots, \hat{\bar{Y}}_j, \dots, \bar{Y}_{q+1}).$$

In particular, we see from the first equality that the operator $A : C^{p,q}(M) \longrightarrow C^{p+2,q-1}(M)$ is of order 0.

Let δ, δ' and A^* respectively denote the adjoint operators of the operators :

$$d : A^{p,q}(M) \longrightarrow A^{p,q+1}(M),$$

$$d' : C^{p,q}(M) \longrightarrow C^{p+1,q}(M),$$

$$A : C^{p,q}(M) \longrightarrow C^{p+2,q-1}(M).$$

Clearly we have $\delta\varphi = \pi_{p,q-1}(A^*\varphi + \delta'\varphi + \delta''\varphi)$ for all $\varphi \in A^{p,q}(M)$, $\pi_{p,q-1}$ being the orthogonal projection : $A^{p+q-1}(M) \longrightarrow A^{p,q-1}(M)$.

Let $x \in M$ and let $(e_1, \ldots, e_{n-1})$ be an orthonormal base of S_x. Put $e_0 = \xi_x$. Then $(e_0, \ldots, e_{n-1})$ gives an orthonormal base of $\hat{T}(M)_x$.

Lemma 8.1. For any $\varphi \in C^{p+1,q}(M)$, $\delta'\varphi$ may be described as follows :

$$(\delta'\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$
$$= - \sum_{\lambda=0}^{n-1} (\nabla_{\bar{e}_\lambda}\varphi)(e_\lambda, X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q).$$

Proof. Define $\varphi' \in C^{p,q}(M)$ by the right hand side of the formula above. For any $\psi \in C^{p,q}(M)$, we have

$$< \varphi, d'\psi > = <\varphi', \psi> + \sum_\lambda (\nabla_{\bar{e}_\lambda} \alpha)(e_\lambda),$$

where α is the cross section of $\hat{T}(M)^*$ defined by $\alpha(X) =$

$< X \lrcorner \varphi, \psi >$ for all $X \in \hat{T}(M)_x$. Since

$$\sum_\lambda (\nabla_{\bar{e}_\lambda} \alpha)(e_\lambda) = \xi\, \alpha(\xi) + \sum_{i=1}^{n-1} (\nabla_{\bar{e}_i} \alpha)(e_i),$$

we see from Proposition 3.6 that the integral of $(\sum_\lambda (\nabla_{\bar{e}_\lambda} \alpha)(e_\lambda))dv$ over M vanishes. Hence $\varphi' = \delta'\varphi$, proving Lemma 8.1.

Lemma 8.2. The operator $\Delta' = \delta'd' + d'\delta' : C^{p,q}(M) \to C^{p,q}(M)$ is strongly elliptic.

Proof. Let $\varphi \in C^{p,q}(M)$. Then we have

$$(\delta'd'\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= - \sum_\lambda (\nabla_{\bar{e}_\lambda} \nabla_{e_\lambda} \varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$+ \sum_{\lambda, i} (-1)^{i+1} (\nabla_{\bar{e}_\lambda} \nabla_{X_i} \varphi)(e_\lambda, X_1, \ldots, \hat{X}_i, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q),$$

$$(d'\delta'\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= \sum_{\lambda, i} (-1)^i (\nabla_{X_i} \nabla_{\bar{e}_\lambda} \varphi)(e_\lambda, X_1, \ldots, \hat{X}_i, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q).$$

Therefore by using the Ricci formula, we obtain

$$\Delta'\varphi = - \sum_\lambda (\nabla_{\bar{e}_\lambda} \nabla_{e_\lambda} \varphi) + W\varphi,$$

where W is an operator of order 1. Since $e_0, \ldots, e_{n-1}, \bar{e}_1, \ldots, \bar{e}_{n-1}$ form a base of $\mathbb{C}T(M)_x$, it follows from this formula that Δ' is strongly elliptic.

Lemma 8.3. The operator $\delta'd'' + d''\delta' : C^{p,q}(M) \longrightarrow C^{p-1,q+1}(M)$ is of order 1.

Proof. Let $\varphi \in C^{p,q}(M)$. Then we have

$$(\delta'd''\varphi)(X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \bar{Y}_{q+1})$$

$$= (-1)^p \sum_{\lambda,j} (-1)^j (\nabla_{\bar{e}_\lambda} \nabla_{\bar{Y}_j} \varphi)(e_\lambda, X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \hat{\bar{Y}}_j,\dots, \bar{Y}_{q+1})$$

$$+ (W_1\varphi)(X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \bar{Y}_{q+1}),$$

$$(d''\delta'\varphi)(X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \bar{Y}_{q+1})$$

$$= (-1)^{p-1} \sum_{\lambda,j} (-1)^j (\nabla_{\bar{Y}_j} \nabla_{\bar{e}_\lambda} \varphi)(e_\lambda, X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \hat{\bar{Y}}_j,\dots, \bar{Y}_{q+1})$$

$$+ (W_2\varphi)(X_1,\dots, X_{p-1}, \bar{Y}_1,\dots, \bar{Y}_{q+1}),$$

where both W_1 and W_2 are operators of order 1. Therefore from the Ricci formula we find that $\delta'd'' + d''\delta'$ is of order 1.

8.2. The finiteness for the groups $H_*^{k-1,1}(M)$ and $H_0^k(M)$. In this paragraph, we assume that $n \geq 3$. Let k be any integer. Now $H_*^{k-1,1}(M)$ was the cohomology group of the complex :

$$A^{k-1,0}(M) \xrightarrow{d} A^{k-1,1}(M) \xrightarrow{d} A^{k-1,2}(M).$$

Let us consider the Laplacian :

$$\Delta = \delta d + d\delta : A^{k-1,1}(M) \longrightarrow A^{k-1,1}(M).$$

We also consider the operator

$$K = \delta'd'' + d''\delta' + A^*d' \quad : C^{k,0}(M) \longrightarrow C^{k-1,1}(M),$$

of which the adjoint operator is

$$K^* = \delta''d' + d'\delta'' + \delta'A : C^{k-1,1}(M) \longrightarrow C^{k,0}(M).$$

Lemma 8.4. Let $\varphi = \varphi_0 + \varphi_1 \in A^{k-1,1}(M)$, where $\varphi_0 \in C^{k,0}(M)$ and $\varphi_1 \in C^{k-1,1}(M)$. Then we have

$$(\Delta\varphi, \varphi) \geq (\Delta'\varphi_0, \varphi_0) + (\Delta''\varphi_1, \varphi_1) + 2\mathrm{Re}(K\varphi_0, \varphi_1).$$

Proof. We have

$$\Delta\varphi_0 = (\Delta'\varphi_0 + \delta''d''\varphi_0) + K\varphi_0,$$

$$\Delta\varphi_1 = K^*\varphi_1 + (\delta'd'\varphi_1 + \Delta''\varphi_1 + A^*A\varphi_1).$$

Hence

$$(\Delta\varphi, \varphi) = (\Delta'\varphi_0, \varphi_0) + (\delta''d''\varphi_0, \varphi_0) + (K\varphi_0, \varphi_1) + (K^*\varphi_1, \varphi_0)$$
$$+ (\delta'd'\varphi_1, \varphi_1) + (\Delta''\varphi_1, \varphi_1) + (A^*A\varphi_1, \varphi_1).$$

Lemma 8.4 is now clear from this equality, because $(K^*\varphi_1, \varphi_0) = \overline{(K\varphi_0, \varphi_1)}$, $(\delta''d''\varphi_0, \varphi_0) = (d''\varphi_0, d''\varphi_0) \geq 0$, etc.

Now the spaces $C^{p,q}(M)$ $(= C^q(M, E^p))$ are equipped with the Sobolev norms $\|\ \ \|_{(s)}$. (See §6.) These norms yield the product norms $\| \ \|_{(s)}$ in the spaces $A^{p,q}(M)$. We also consider the norm $\| \ \|$ in $A^{p,q}(M)$ defined by $\|\varphi\|^2 = (\varphi, \varphi)$.

Theorem 8.5. The operator $\Delta : A^{k-1,1}(M) \longrightarrow A^{k-1,1}(M)$ is subelliptic or more precisely

$$\|\varphi\|^2_{(\frac{1}{2})} \leq C((\Delta\varphi, \varphi) + \|\varphi\|^2), \quad \varphi \in A^{k-1,1}(M).$$

Proof. Since Δ' is a self-adjoint, strongly elliptic operator (Lemma 8.2), we have

$$C_1\|\varphi_0\|^2_{(1)} \leq (\Delta'\varphi_0, \varphi_0) + \|\varphi_0\|^2, \quad \varphi_0 \in C^{k,0}(M).$$

Since $n \geq 3$, it follows from Theorem 6.3 that

$$C_2\|\varphi_1\|^2_{(\frac{1}{2})} \leq (\Delta''\varphi_1, \varphi_1) + \|\varphi_1\|^2, \quad \varphi_1 \in C^{k-1,1}(M).$$

The operator K is of order 1 by Lemma 8.3 and hence

$$\|K\varphi_0\| \leq C_3\|\varphi_0\|_{(1)}.$$

Furthermore, for any positive number ε, we have

$$-\varepsilon\|K\varphi_0\|^2 - \frac{1}{\varepsilon}\|\varphi_1\|^2 \leq \mathrm{Re}(K\varphi_0, \varphi_1).$$

From these inequalities and Lemma 8.4 follows that

$$(C_1 - 2\varepsilon C_3^2)\|\varphi_0\|^2_{(1)} + C_2\|\varphi_1\|^2_{(\frac{1}{2})}$$
$$\leq (\Delta\varphi, \varphi) + (1 + \tfrac{2}{\varepsilon})\|\varphi\|^2.$$

Thus, choosing ε such that $C_1 - 2\varepsilon C_3^2 > 0$, we get an equality of the form in Theorem 8.5.

Above all we see from Theorem 8.5 that Δ is hypoelliptic (cf. Theorem 6.4). Thus, as in §6, we have the following :

1°. $\mathcal{H}_*^{k-1,1}(M) = \{ \varphi \in A^{k-1,1}(M) \mid \Delta\varphi = 0 \}$, the space of harmonic forms, is finite dimensional ;

2° $H_*^{k-1,1}(M) \cong \mathcal{H}_*^{k-1,1}(M)$.

Therefore, using Proposition 1.2, we get

Theorem 8.6. For any k, we have

$$\dim H_0^k(M) \leq \dim H_*^{k-1,1}(M) < \infty .$$

Remark. Since $H^{p,q}(M) = E_1^{p,q}(M)$ and $H_0^k(M) = E_2^{k,0}(M)$, a formal argument on the filtration $\{ F^p(A^k(M))\}$ proves the inequality :

$$\dim H_0^k(M) \leq \dim H^k(M) + \sum \dim H^{p,q}(M),$$

where the sum $\sum$ is taken over all the pairs (p, q) with $p + q = k-1$, $p \geq 0$ and $q \geq 1$. This inequality combined with (1) of Theorem 6.5 implies that $H_0^k(M)$ is finite dimensional if $k \neq n$.

§9. Differentiable families of compact strongly pseudo-convex manifolds.

9.1. The upper semi-continuity for $\dim H^{p,q}(M)$ and $\dim H_*^{k-1,1}(M)$. Let Ω be a domain of the space $\mathbb{R}^\ell$ of ℓ real variables, and $\{M_t\}_{t\in\Omega}$ a family of compact s. p. c. manifolds parametrized by Ω. Then the family $\{M_t\}$ is said to be differentiable if there is a fibred manifold $\mathcal{M}$ over Ω with projection π which satisfies the following conditions :

1) The projection π is proper ;

2) For every $t \in \Omega$, $\pi^{-1}(t) = M_t$ as differentiable manifolds ;

3) Let S_t be the s. p. c. structure of M_t, where we note that $S_t \subset \mathbb{C}T(M_t) \subset \mathbb{C}T(\mathcal{M})$. Then the union $\bigcup_{t\in\Omega} S_t$ gives a differentiable subbundle of $\mathbb{C}T(\mathcal{M})$.

Theorem 9.1. Let $\{M_t\}_{t\in\Omega}$ be a differentiable family of compact s. p. c. manifolds of dimension $2n - 1 \geq 5$. Then the functions

$$\Omega \ni t \longrightarrow \dim H^{p,q}(M_t) \in \mathbf{Z}, \quad q \neq 0, n-1,$$

and the functions

$$\Omega \ni t \longrightarrow \dim H_*^{k-1,1}(M_t) \in \mathbf{Z}$$

are all upper semi-continuous, where $\mathbf{Z}$ denotes the set of integers equipped with the discrete topology.

It should be noted that the functions

$$\Omega \ni t \longrightarrow \dim H_0^k(M) \in \mathbf{Z}$$

do not have upper nor lower semi-continuity in general.

Proof of Theorem 9.1. Take any $t_0 \in \Omega$ and let us work around t_0. Since the projection $\pi : M \to \Omega$ is proper, the fibred manifold M is locally trivial. Therefore we may assume that, for each $t \in \Omega$, $M_t = M_{t_0}$ as differentiable manifolds. In this case the differentiability for the family $\{M_t\}$ means that $\{S_t\}$ is a differentiable family of subbundles of $\mathbb{C}T(M_{t_0})$. Let P_t be the real part of $S_t + \bar{S}_t$. Then $\{P_t\}$ gives a differentiable family of contact structures on M_{t_0}, which is locally trivial by Martinet

[17]. Thus we may further assume that $P_t = P_{t_0}$ for each $t \in \in \Omega$.

We take a basic field ξ for the central s. p. c. manifold M_{t_0} which is simultaneously a basic field for every s. p. c. manifold M_t. Starting with the s. p. c. manifold M_t and the basic field ξ , we difine the inner products $(\ , \ )$ in $C^{p,q}(M_t)$, the operators $\Delta'' : C^{p,q}(M_t) \longrightarrow C^{p,q}(M_t)$, etc. just as in §7, which will be written as $(\ , \ )_t$, Δ''_t, etc. We also consider the Sobolev norms $\| \ \|_{(s)}$ in $C^{p,q}(M_{t_0})$. If we choose Ω sufficiently small, we can find a differentiable family of (base preserving) bundle automorphisms of $T(M_{t_0})$, $\{\tau_t\}$, such that τ_{t_0} is the identity and $\tau_t S_t = S_{t_0}$. Each τ_t induces the bundle isomorphisms : $C^{p,q}(M_t) \longrightarrow C^{p,q}(M_{t_0})$ in a natural manner , which we shall denote by the same symbol τ_t. The norms $\|\tau_t \varphi\|_{(s)}$, $\varphi \in C^{p,q}(M_t)$, will be denoted simply by $\| \ \|_{(s)}$.

Let K be any compact subset of Ω .

Lemma 9.2. If $q \neq 0, n - 1$, we have

$$\| \varphi \|^2_{(\frac{1}{2})} \leq C((\Delta''_t \varphi, \varphi)_t + (\varphi, \varphi)_t),$$

$$\varphi \in C^{p,q}(M_t), \quad t \in K.$$

Remark that this estimation is uniform with respect to the parameter t, i.e., the constant C does not depend on t. The proof of Lemma 9.2 is easy from those of Propositions 6.1 and 6.2 if we note that the estimation in Proposition 6.2 essentially depends on the contact structure P only.

Let us now consider the operators $\Delta_t : A^{k-1,1}(M_t) \longrightarrow A^{k-1,1}(M_t)$ Then from Lemma 9.2 and the proof of Theorem 8.5, we can easily obtain the following

Lemma 9.3.

$$\| \varphi \|^2_{(\frac{1}{2})} \leq C((\Delta_t \varphi, \varphi)_t + (\varphi, \varphi)_t),$$

$$\varphi \in A^{k-1,1}(M_t), \quad t \in K.$$

Now we have established Lemmas 9.2 and 9.3 , we are able to prove Theorem 9.1 in a standard fashion, using the Rellich lemma and the hypoellipticity for Δ''_t and Δ_t (cf. Kodaira-Spencer [11]).

9.2. A remark on holomorphic imbeddings. Let M be a s. p. c. manifold, and $\mathbb{f}$ an imbedding of M in $\mathbb{C}^N$. Then we say that the imbedding $\mathbb{f}$ is holomorphic or the s. p. c. manifold M is realized as a real submanifold in $\mathbb{C}^N$ by $\mathbb{f}$, if the s. p. c. structure S of M is induced from $\mathbb{f}$. Let $\mathbb{f} = (f^1, \ldots, f^N)$. Then it is easy to see that the imbedding $\mathbb{f}$ is holomorphic if and only if each component f^i of $\mathbb{f}$ is a holomorphic function on the s. p. c. manifold.

Recently Harvey-Lawson [5] proves that if M is compact and if $\mathbb{f} : M \longrightarrow \mathbb{C}^N$ is holomorphic, then the image $\mathbb{f}(M)$ of M by $\mathbb{f}$ is the boundary of a uniquely determined subvariety of $\mathbb{C}^N - M$, a solution of the Plateaux problem.

Theorem 9.4. Let $\{M_t\}_{t\in\Omega}$ be a differentiable family of compact s. p. c. manifolds of dimension $2n-1 \geq 5$ and let $\pi : \mathcal{M} \longrightarrow \Omega$ be the associated fibred manifold. Asssume that, for some t_0, there is given a holomorphic imbedding $\mathbf{g} : M_{t_0} \longrightarrow \mathbb{C}^N$ and that $\dim H^{0,1}(M_t)$ is constant in a neighborhood of t_0. Then there exist a neighborhood U of t_0 and an imbedding $\mathbb{F} : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^N$ satisfying the following conditions :

1) $\tilde{\omega} \circ \mathbf{F} = \pi \mid \pi^{-1}(U)$, $\tilde{\omega}$ being the projection of $U \times \mathbb{C}^N$ onto U. In other words, if $t \in U$ and $p \in M_t$, then $\mathbf{F}(p)$ is of the form : $(t, \mathbf{f}_t(p))$.

2) For each $t \in U$, the assignment $p \longrightarrow \mathbf{f}_t(p)$ gives a holomorphic imbedding of the s. p. c. manifold M_t in $\mathbb{C}^N$.

3) $\mathbf{f}_{t_0} = \mathbf{g}$.

Outline of the proof. As in the proof of Theorem 9.1, we may assume that $M_t = M_{t_0}$ (as differentiable manifolds) and $P_t = P_{t_0}$. Let us consider the operators $\Delta''_t : C^{0,1}(M_t) \longrightarrow C^{0,1}(M_t)$.

Lemma 9.5. Let K be any compact subset of Ω. Then, for any non-negative interger m, there is a constant C_m such that

$$\| \varphi \|_{(m+\frac{1}{2})} \leq C_m \| \Delta''_t \varphi + \varphi \|_{(m-\frac{1}{2})} ,$$

$$\varphi \in C^{0,1}(M_t), \quad t \in K.$$

We can prove this fact in the same manner as Kohn [13], based on Lemma 9.2 or properly uniform estimations for the operators Δ''_t corresponding to Propositions 6.1 and 6.2. From Lemma 9.5 follows the inequalities :

$$(9.1)\quad \|\varphi\|^2_{(m+\frac{1}{2})} \leq C'_m (\|\Delta''_t\varphi\|^2_{(m)} + \|\varphi\|^2_{(0)}),$$

$$\varphi \in C^{0,1}(M_t), \quad t \in K.$$

Lemma 9.6. In a small neighborhood V of t_0, the Green operator G_t of the operator $\Delta''_t : C^{0,1}(M_t) \longrightarrow C^{0,1}(M_t)$ differentiably depends on the parameter t, that is, if $\{\varphi_t\}_{t\in V}$ is a differentiable family of elements φ_t of $C^{0,1}(M_t)$ (i.e., $(\tau_t\varphi_t)_p$ is C^∞ in the two variables t and p), so is the family $\{G_t\varphi_t\}$.

The proof of this fact is based on (9.1), and is quite similar to that of Kodaira-Spencer [11], Theorem 5.

We are now in position to prove Theorem 9.4. Let us consider the operators $d''_t : C^{0,q}(M_t) \longrightarrow C^{0,q+1}(M_t)$, $\delta''_t : C^{0,q}(M_t) \longrightarrow C^{0,q-1}(M_t)$ and $H_t : C^{0,0}(M_t) \longrightarrow C^{0,0}(M_t)$. ($H_t$ was defined in 6.2.) We express $\mathbf{g}$ as $(g^1, .., g^N)$ and put

$$f^i_t = H_t g^i = g^i - \delta''_t G_t d''_t g^i, \quad t \in V, \quad 1 \leq i \leq N,$$

which is a holomorphic function on the s.p.c. manifold M_t. Since G_t differentiably depends on t (Lemma 9.6), so is f_t^i. Since $d''_{t_0} g^i = 0$, we have $f_{t_0}^i = g^i$. Now, for each t, define a map $\mathbf{f}_t : M_t \longrightarrow \mathbb{C}^N$ by $\mathbf{f}_t = (f_t^1, \ldots, f_t^N)$. Then $\mathbf{f}_t$ differentiably depends on t and $\mathbf{f}_{t_0} = \mathbf{g}$. Since $\mathbf{g}$ is an imbedding, we can find a neighborhood U of t_0 such that $\mathbf{f}_t$ is an imbedding for each $t \in U$. We have thus proved Theorem 9.4.

Remark. Let M be a compact, connected, contact manifold of dimension $2n - 1$, and let P be its contact structure. We denote by $\mathcal{S}(M, P)$ the set of all s.p.c. structures S on M such that the contact structure associated to S is just equal to the given P. For $S \in \mathcal{S}(M, P)$, we denote by M_S the s.p.c. manifold with the s.p.c. structure S.

Let G^{n-1} be the set of all complex contact elements of dimension $n-1$ to M, which is a fibre bundle over M. Then an $(n-1)$-dimensional subbundle of $\mathbb{C}T(M)$ gives a cross section of G^{n-1}, and vice versa. In particular $\mathcal{S}(M, P)$ may be regarded as a subset of $\Gamma(G^{n-1})$, the set of all cross sections of G^{n-1}, and hence we have the notion of the C^{ℓ}-topology in $\mathcal{S}(M, P)$.

Let N be an integer with $N \geq n$. Let $\mathcal{S}^N(M, P)$ be the set of all $S \in \mathcal{S}(M, P)$ such that the s.p.c. manifold M_S can be realized as a real submanifold in $\mathbb{C}^N$. Then it can be proved that the set $\{ S \in \mathcal{S}^N(M, P) \mid H^{0,1}(M_S) = 0 \}$ is an open set of $\mathcal{S}(M, P)$ w. r. t. the C^{n+4}-topology (cf. Theorem 9.4). Note that every $S \in \mathcal{S}^n(M, P)$ satisfies $H^{0,1}(M_S) = 0$ by Theorem 10.3.

Now consider the case where M is the unit sphere S^{2n-1} in $\mathbb{C}^n$. Let S_0 be the s.p.c. structure induced from the injection $S^{2n-1} \longrightarrow \mathbb{C}^n$ and let P_0 be the associated contact sturcture. Then we raise the question : Is it true that $\mathcal{S}^n(S^{2n-1}, P_0) = \mathcal{S}(S^{2n-1}, P_0)$?

The Brieskorn variety B^5 of type $(2,2,2,3)$ is known to be diffeomorphic with the 5-sphere S^5 (cf. Milnor [18]) and there is defined on B^5 a s.p.c. structure S_1 (see 11.2). This will imply that there is a s.p.c. structure on S^5 which can not be realized as a real hypersurface in $\mathbb{C}^3$. However we do not know whether the contact structure P_1 associated to S_1 is equivalent to the "standard" cantact structure P_0 or not.

§10. Strongly pseudo-convex manifolds and isolated singular points

10.1. The boundaries of Stein manifolds. Let M' be an n-dimensional complex manifold, where $n \geq 3$, and V a relatively compact domain of M'. Assume the following : 1) The boundary $M = \partial V$ is smooth and connected ; 2) M is s.p.c. in M'; 3) V lies inside M (see 2.3).

Let us consider the complex manifold with boundary, $\bar{V} = V \cup M$. We denote by $A^k(\bar{V})$ the restriction of $A^k(M')$ to $\bar{V}$ and by $A^k(\bar{V})$ the space of C^∞ cross sections of $A^k(\bar{V})$. ($A^k(\bar{V})$ is nothing but the space of all the restrictions $\varphi|\bar{V}$ of $\varphi \in A^k(M')$ to $\bar{V}$). The exterior differentiation $d : A^k(M') \longrightarrow A^{k+1}(M')$ induces the operator $d : A^k(\bar{V}) \longrightarrow A^{k+1}(\bar{V})$. Thus we obtain the complex $\{ A^k(\bar{V}), d \}$, the de Rham complex of $\bar{V}$. We denote by $H^k(\bar{V})$ the cohomology groups of this complex. It is known that $H^k(\bar{V})$ is isomorphic with the k-th cohomology group of $\bar{V}$ with complex coefficients (e.g. Nagano [20]).

We denote by $F^p(A^k(\bar{V}))$ the restriction of $F^p(A^k(M'))$ to $\bar{V}$ and by $F^p(A^k(\bar{V}))$ the space of C^∞ cross sections of $F^p(A^k(\bar{V}))$. Then the collection $\{F^p(A^k(\bar{V}))\}$ gives a filtration of the de Rham complex $\{A^k(\bar{V}), d\}$ and we denote by $\{E_r^{p,q}(\bar{V})\}$ the spectral sequence associated with the filtration.

The injection $\iota_M : M \longrightarrow \bar{V}$ induces, as usual, the morphism of complexes

$$\iota_M^* : \{ A^k(\bar{V}), d\} \longrightarrow \{ A^k(M), d\},$$

which clearly preserves the filtrations, i.e., $\iota_M^* F^p(A^k(\bar{V})) \subset F^p(A^k(M))$. Thus ι_M induces the morphism of spectral sequences

$$\iota_M^* : E_r^{p,q}(\bar{V}) \longrightarrow E_r^{p,q}(M).$$

In the same way the injection $\iota_V : V \to \bar{V}$ induces the morphism

$$\iota_V^* : E_r^{p,q}(\bar{V}) \longrightarrow E_r^{p,q}(V).$$

Lemma 10.1. (1) (Kohn [12] and Hörmander [8].) The map $\iota_V^* : E_1^{p,q}(\bar{V}) \longrightarrow E_1^{p,q}(V)$ is an isomorphism for any (p, q) with $q \neq 0$.

(2) (Kohn-Rossi [15] and Folland-Kohn [2])

$$\dim E_1^{p,q}(M) \leq \dim E_1^{p,q}(\bar{V}) + \dim E_1^{n-p,n-q-1}(\bar{V})$$

for any (p, q) with $q \neq 0, n-1$.

To accept this lemma, we must observe the following : 1°. $E_1^{p,q}(V)$ (resp. $E_1^{p,q}(\bar{V})$) are the Dolbeault cohomology groups of the complex manifold V (resp. of the complex manifold with boundary $\bar{V}$); 2°. $E_1^{p,q}(M)$ are the boundary cohomology groups $H^{p,q}(\mathcal{B})$ of the boundary $M = \partial V$ introduced by Kohn - Rossi [13].

Lemma 10.2. If M' is a Stein manifold, then the map $\iota_M^* : E_1^{k,0}(\bar{V}) \longrightarrow E_1^{k,0}(M)$ is an isomorphism for any k.

This fact follows immediately from Theorem 3-5 of Shiga [26] (cf. Kohn - Rossi [15]), if we remark the following : 1°. The holomorphic vector bundle $\Lambda^k\hat{T}(M)^*$ over the s.p.c. manifold M is the restriction of the holomorphic vector bundle $\Lambda^k\hat{T}(M')^*$ over the complex manifold M' to M ; 2°. $E_1^{k,0}(\bar{V})$ is the space of C^∞ cross sections of $\Lambda^k\hat{T}(M')^*|\bar{V}$ which are holomorphic, restricted to

V ; 3°. $E_1^{k,0}(M)$ is the space of C^∞ holomorphic cross sections of $\Lambda^k\hat{T}(M)^*$.

Using Lemmas 10.1 and 10.2, we shall now prove the following

Theorem 10.3. If M' is a Stein manifold, then we have :

(1) $H^{p,q}(M) = 0$ for any (p, q) with $q \neq 0, n-1$.

(2) $H_0^k(M) \cong H^k(V)$ for any k.

Proof. M' being a Stein manifold, V is also a Stein manifold and hence $E_1^{p,q}(V) = 0$ for $q \neq 0$ (e.g. Gunnning - Rossi [4]). Therefore (1) is clear from Lemma 10.1. Let k be any integer. By Lemma 10.2, we see that the map $\iota_M^* : E_2^{k,0}(\bar{V}) \longrightarrow E_2^{k,0}(M)$ is an isomorphism. By using (1) of Lemma 10.1, we have $E_1^{p,q}(\bar{V}) \cong E_1^{p,q}(V) = 0$ for $q \neq 0$. It follows that the (natural) map $E_2^{k,0}(\bar{V}) \to H^k(\bar{V})$ is an isomorphism. Furthermore the map ι_V^* : $H^k(\bar{V}) \longrightarrow H^k(V)$ is clearly an isomorphism. Thus we have shown that $E_2^{k,0}(M) \cong H^k(V)$, completing the proof of Theorem 10.3.

Corollary. The assumption being as in Theorem 10.3, we have :

(1) The (natural) map $H_0^k(M) \to H^k(M)$ is an isomorphism

for any $k \leq n - 2$.

(2) The map $H_0^{n-1}(M) \longrightarrow H^{n-1}(M)$ is injective.

In the next paragraph we shall treat the case where V admits (isolated) singular points.

10.2. Isolated singlar points of complex hypersurfaces.

We first prove the following

Proposition 10.4. Let $\{M_t\}_{t\in\Omega}$ be a differentiable family of compact s.p.c. manifolds of dimension $2n - 1 \geq 5$. Given intergers k and μ, let $\Omega(k, \mu)$ denote the subset of Ω consisting of all t which satisfy $H^{k-1,1}(M_t) = 0$ and $\dim H_0^k(M_t) = \mu$. If $\Omega(k, \mu) \neq \phi$, then we have the inequalities :

$$\mu \leq \dim H_*^{k-1,1}(M_{t_0}) \leq \dim H^{k-1,1}(M_{t_0}) + \dim H_0^k(M_{t_0}) \quad \text{for every}$$

t_0 in the closure of $\Omega(k, \mu)$ (in Ω).

Proof. Let $t \in \Omega(k, \mu)$. Since $H^{k-1,1}(M_t) = 0$, it follows from Proposition 1.2 that $H_*^{k-1,1}(M_t) \cong H_0^k(M_t)$. Hence $\dim H_*^{k-1,1}(M_t) = \mu$ for any $t \in \Omega(k, \mu)$. Therefore we see from Theorem 9.1 that $\dim H_*^{k-1,1}(M_{t_0}) \geq \mu$ for every t_0 in the closure of $\Omega(k, \mu)$. Proposition 10.4 now follows from this fact and Proposition 1.2.

We shall apply Proposition 10.4 to the study of isolated singular points of complex hypersurfaces.

Let $f(z_1, \dots, z_{n+1})$ be a polynomial function on $\mathbb{C}^{n+1}$, where $n \geq 3$. Assume that $f(0) = 0$ and that there is a neighborhood U of the origin 0 of $\mathbb{C}^{n+1}$ such that the differential df_z does not vanish at each $z \in U - \{0\}$.

Let $S^{2n+1}(r)$ (resp. $B^{2n+2}(r)$) be the sphere (resp. the open ball) in $\mathbb{C}^{n+1}$ of radius r centred at the origin, and let ζ be the real polynomial function on $\mathbb{C}^{n+1}$ defined by $\zeta(z) = \sum_i |z_i|^2$. Now consider a small ε with $B^{2n+2}(\varepsilon) \cup S^{2n+1}(\varepsilon) \subset U$, and let M (resp. V) be the intersection of the complex hypersurface $f^{-1}(0)$ with the sphere $S^{2n+1}(\varepsilon)$ (resp. with the open ball $B^{2n+2}(\varepsilon)$). Clearly V is a relatively compact open set of $f^{-1}(0)$ and $M = \partial V$. Milnor [18] proves that, for ε sufficiently small, M is a smooth real hypersurface of the complex hypersurface $f^{-1}(0)$, that is, the differentials df_z, $d\bar{f}_z$ and $d\zeta_z$ are linearly independent over $\mathbb{C}$ at each $z \in M$. (He also shows that both M and V are connected.) Fix such an ε from now on.

We assert that M is s.p.c. in $f^{-1}(0)$ and V lies inside $M = \partial V$. Indeed let ζ_1 be the restriction of ζ to $f^{-1}(0)$. Then M (resp. V) is defined by $\zeta_1 = \varepsilon^2$ (resp. by $\zeta_1 < \varepsilon^2$) and the quadratic form $L(\zeta_1)_z(X, \bar{X}) = \sum_i |Xz_i|^2 \quad X \in S(\zeta_1)_z$, is positive definite at each $z \in M$, proving our assertion (see 2.3). We have thus known that V is a s.p.c. domain in the complex hypersurface $f^{-1}(0)$ with a single isolated singular point, the origin (cf. Gunning - Rossi [4]).

Theorem 10.5. Let μ be the Milnor number (or the multiplicity) of the isolated singular point ([18]). Then we have the inequality

$$\mu \leq \dim H^{n-1,1}(M) + \dim H_0^n(M).$$

Proof. Let Ω be a small open disk in $\mathbb{C}$ centred at the origin. We define an open set $\mathcal{M}$ of $S^{2n+1}(\varepsilon)$ by $\mathcal{M} = f^{-1}(\Omega) \cap S^{2n+1}(\varepsilon)$, and consider the proper map $\pi : \mathcal{M} \ni z \longrightarrow f(z) \in \Omega$. Now the fact that the differentials df_z, $d\bar{f}_z$ and $d\zeta_z$ are linearly independent at each $z \in M = \pi^{-1}(0)$, means that the map $S^{2n+1}(\varepsilon) \ni z \to f(z) \in \mathbb{C}$ is of maximum rank at each $z \in M$. Therefore for Ω sufficiently small, we have : 1°. M is a fibred manifold over the base space Ω with projection π ; 2°. Each fibre $M_t = \pi^{-1}(t)$ is a s.p.c. real hypersurface of the complex hypersurface $f^{-1}(t)$. Furthermore if we put $V_t = f^{-1}(t) \cap B^{2n+2}(\varepsilon)$, $t \in \Omega$, we see that V_t is a relatively compact domain of $f^{-1}(t)$ and lies inside $M_t = \partial V_t$. The notation being as in Proposition 10.4, we now assert that $\Omega - \{0\} \subset \Omega(n, \mu)$. Indeed, let $t \in \Omega - \{0\}$. Then we have $H^{n-1,1}(M_t) = 0$ and $H_0^n(M_t) \cong H^n(V_t)$ by Theorem 10.3. (Note that there is a neighborhood M'_t of $V_t \cup M_t$ in $f^{-1}(t)$ such that M'_t is a Stein manifold.) On the other hand, Milnor [18] proves that $\mu = \dim H^n(V_t)$. We have thus shown that $t \in \Omega(n, \mu)$, proving our assertion. Theorem 10.5 is now immediate from Proposition 10.4.

Remarks. (1) Let $t \in \Omega - \{0\}$. Milnor [18] proves that

$$\begin{cases} H^0(V_t) \cong \mathbb{C}, \\ H^k(V_t) = 0, \quad 1 \le k \le n-1. \end{cases}$$

It follows from Theorem 10.3 that

$$\begin{cases} H_0^0(M_t) \cong \mathbb{C}, \\ H_0^k(M_t) = 0, \quad 1 \le k \le n-1. \end{cases}$$

(2) There naturally arised the question of whether, in Theorem 10.5, equality " = " holds in general or not. Concerning this question, Naruki has recently succeeded in obtaining the exact expression of the Milnor number μ. The result is

(N. 1) $\mu = \dim H^{n-1,1}(M) + \dim H_0^n(M) - \dim H_0^{n-1}(M)$

(cf. [21] and [22]). For the details, see the forthcoming papers of Naruki.

Note that this equality remains valid even if M is replaced by M_t, showing that the Milnor number μ is an invariant of the family $\{M_t\}_{t\in\Omega}$.

Naruki has also proved the following facts :

(N. 2) $H^{p,q}(M) = 0$ for any pair (p, q) with $p + q \neq n - 1$,

n and $q \neq 0, n-1$, and the groups $H^{p,q}(M)$, where $p+q = n-1$ or n and $1 \leq q \leq n-2$, are mutually isomorphic ;

$$\text{(N. 3)} \quad \begin{cases} H_0^0(M) \cong \mathbf{C}, \\ H_0^k(M) = 0, \quad 1 \leq k \leq n-2 . \end{cases}$$

Note that (N. 3) can be also derived from (N. 2) and the fact due to [18] : $H^0(M) \cong \mathbf{C}$ and $H^k(M) = 0$, $1 \leq k \leq n-2$.

(3) The Ricci operator. For example consider the polynomial function $f(z) = z_1^2 + \ldots + z_n^2 + z_{n+1}^3$ (cf. Milnor [18]). Then the origin 0 is the only isolated singular point of $f^{-1}(0)$ and, for any $\varepsilon > 0$, the intersection $M_\varepsilon = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is a compact s.p.c., real hypersurface of $f^{-1}(0)$ (see 11.2). Let ξ be the basic field on M_ε corresponding to the basic form $\theta = \sqrt{-1} \sum_i \varphi_i d\overline{\varphi}_i$, φ_i being the restriction of z_i to M_ε. If $\varepsilon \geq \sqrt{2}$, then it can be proved that the Ricci operator R_* associated to the pair (M_ε, ξ) is positive definite everywhere. By Proposition 7.4 it follows that $H^{0,q}(M_\varepsilon) = 0$, $1 \leq q \leq n-2$, and this fact reproduces Naruki's result (N. 2) in this special case.

In general (N. 2) can be easily obtained, once we have established

(N. 2') $\quad H^{0,q}(M) = 0, \quad 1 \leq q \leq n-2.$

(This is based on the fact that both $E = \hat{T}(\mathbb{C}^{n+1})|M$ and $E/\hat{T}(M)$ are holomorphically trivial.)

(4) In an analogous way to the proof of Theorem 10.5, we shall be able to apply Proposition 10.4 combined with Theorem 10.3 to the study of more general types of isolated singularities by considering appropriate deformations of the singularities. (Before proceeding to the applications, it will be first necessary to generalize Proposition 10.4 so that the parameter space Ω will be allowed to have singularities.) Thus we shall obtain certain results on the singularities which will generalize Theorem 10.5. We want to take up this problem at another occasion.

III. Normal strongly pseudo-convex manifolds

§11. Normal strongly pseudo-convex manifolds

11.1. Normal s.p.c. manifolds. Let M be a s.p.c. manifold. Let S be the s.p.c. structure of M, and (P, I) its real expression. Recall that a vector field X on M is analytic if it leaves the structure S invariant or $[X, \Gamma(S)] \subset \Gamma(S)$. This condition is equivalent to the following two conditions : 1) X is an infinitesimal contact transformation on the underlying contact manifold, 2) X leaves I invariant or $[X, IY] = I[X, Y]$ for all $Y \in \Gamma(P)$. We also note that X is analytic if and only if the image of X by the natural injective map $T(M) \longrightarrow \hat{T}(M)$ is a holomorphic cross section of $\hat{T}(M)$.

We say that M or the pair (M, ξ) is a normal s.p.c. manifold if M admits an analytic basic field ξ .

Remark. It is known that the Lie algebra $\mathfrak{a}(M)$ of all analytic vector fields on M is of finite dimension $\leq n^2 + 2n$, where $\dim M = 2n - 1$ (Tanaka [27] and [28]). Therefore if M is normal and compact, the set of all analytic basic fields is endowed with the structure of a finite dimensional manifold as an open set of $\mathfrak{A}(M)$.

Let (M, ξ) be a normal s.p.c. manifold. Then the basic field ξ leaves invariant the associated tensor fields θ, ω, I, g and $h(= g + \theta^2)$. It follows that the canonical affine connection ∇ is invariant by ξ, i.e.,

$$[\xi, \nabla_X Y] = \nabla_{[\xi, X]} Y + \nabla_X [\xi, Y], \qquad X, Y \in \Gamma(T(M)).$$

We have

$$T_\xi = -\frac{1}{2} I L_\xi I = 0$$

and hence

$$\nabla_\xi Y = L_\xi Y.$$

Therefore the curvature R satisfies

$$R(\xi, Y) = 0.$$

Furthermore by using Proposition 3.5, we can verify the equality

$$R(IX, IY) = R(X, Y).$$

We remark that the collection $\{I, \xi, \theta, h\}$ gives a normal contact metric structure due to Sasaki [25]. Accordingly the results on normal contact metric structures are applicable to our study on normal s.p.c. manifolds.

Here are two important classes of normal s.p.c. manifolds:

Class (I): The class of normal s.p.c. manifolds (M, ξ) such that ξ is induced from a $U(1)$-action, i.e., the toroidal group $U(1)$ differentiably acts on M (in the right) and ξ is induced from the 1-parameter group of transformations : $M \times \mathbb{R} \ni (x, t) \longrightarrow x \cdot e^{t\sqrt{-1}} \in M$;

Class (II) : The class of normal s.p.c. manifolds (M, ξ) such that ξ is induced from a $U(1)$-action and such that the $U(1)$-action is free.

For example, the unit sphere S^{2n-1} in $\mathbb{C}^n$ is a normal s.p.c. manifold entering class (II), where the analytic basic field ξ is induced from the $U(1)$-action : $S^{2n-1} \times U(1) \ni (x, a) \longrightarrow ax \in S^{2n-1}$.

11.2. Weighted homogeneous polynomials. Let $f(z_1, \ldots, z_{n+1})$ be a weighted homogeneous polynomial of type $(a_1, \ldots, a_{n+1})$, where $a_1, \ldots, a_{n+1}$ are positive rational numbers (Milnor [18]). By definition the polynomial f satisfies the equality

$$(11.1) \qquad f(e^{\frac{c}{a_1}} z_1, \ldots, e^{\frac{c}{a_{n+1}}} z_{n+1}) = e^c f(z_1, \ldots, z_{n+1})$$

for every complex number c. Clearly we have $f(0) = 0$. We assume that the origin 0 is an isolated critical point of f. It is then easy to see that the origin is the only isolated critical point of f.

We put $M' = f^{-1}(0) - \{0\}$.

Lemma 11.1. Let $z \in M'$.

(1) $\displaystyle\sum_i \frac{1}{a_i} \frac{\partial f}{\partial z_i}(z)\, z_i = 0.$

(2) The differentials df_z, $d\bar{f}_z$ and $d\zeta_z$ are linearly independent over $\mathbb{C}$.

Proof. By differentiating the both sides of (11. 1) in the variable c at $c = 0$, we have $\sum_i \frac{1}{a_i} \frac{\partial f}{\partial z_i}(z) z_i = f(z) = 0$. This proves (1). Suppose that we have a linear relation :

$$\alpha df_z + \beta d\bar{f}_z + \gamma d\zeta_z = 0,$$

where $\alpha, \beta, \gamma \in \mathbf{C}$. Since $df = \sum_i \frac{\partial f}{\partial z_i} dz_i$ and $d\zeta = \sum_i \bar{z}_i dz_i + \sum_i z_i d\bar{z}_i$, it follows that $\alpha \frac{\partial f}{\partial z_i}(z) + \gamma \bar{z}_i = 0$, $1 \leq i \leq n+1$. Therefore by (1) we obtain $\gamma \sum_i \frac{1}{a_i} |z_i|^2 = 0$, whence $\gamma = 0$. Since $df_z \neq 0$, we have $\alpha = 0$. In the same way we get $\beta = 0$, proving (2).

For every positive number r, let $M(r)$ be the intersection of the complex hypersurface $f^{-1}(0)$ with the sphere $S^{2n+1}(r)$. Then we see from Lemma 11.1 that $M(r)$ is a s.p.c. real hypersurface of M' (cf. 10. 2).

We shall now show that the s.p.c. manifold $M(r)$ is normal. Define a 1-parameter group of holomorphic transformations of $\mathbb{C}^{n+1}$, $\{\tau_t\}$, by

$$\tau_t(z) = (z_1', \ldots, z_{n+1}'), \quad z_i' = e^{\frac{t\sqrt{-1}}{a_i}} \cdot z_i .$$

Clearly $\{\tau_t\}$ leaves invariant M', $S^{2n+1}(r)$ and hence $M(r)$. Let ξ be the vector field on $M(r)$ induced from the 1-parameter group $\{\tau_t \mid M(r)\}$.

Lemma 11.2. The vector field ξ is an analytic basic field on the s.p.c. manifold $M(r)$.

Proof. Let φ_i be the restriction of z_i to $M(r)$. We define a 1-form θ on $M(r)$ by

$$\theta = \frac{\sqrt{-1}}{\lambda} \sum_i \varphi_i \, d\, \bar{\varphi}_i,$$

where λ is the positive function on $M(r)$ defined by

$$\lambda = \sum_i \frac{1}{a_i} |\varphi_i|^2.$$

If we denote by ι the injection $M(r) \longrightarrow \mathbb{C}^{n+1}$, we have $\sum_i \varphi_i d\, \bar{\varphi}_i = \iota^* d'' \zeta$. Hence θ is a basic form on $M(r)$ (see 2.3). Now it is clear that ξ is analytic. Since $\xi \varphi_i = \frac{\sqrt{-1}}{a_i} \varphi_i$, we see easily that ξ is the basic field corresponding to the basic form θ, proving Lemma 11.2.

Note that the normal s.p.c. manifold $(M(r), \xi)$ or exactly $(M(r), c\xi)$ with some positive rational number c enters class (I), because a_i are rational numbers. Consider the special case where $f(z_1, \ldots, z_{n+1}) = \sum_i (z_i)^{a_i}$, a_i being integers ≥ 2. Then the manifold $M(r)$ is well known as a Brieskorn manifold of type

$(a_1, \ldots, a_{n+1})$.

11.3. Normal s.p.c. manifolds entering class (II). Let (M, ξ) be a compact, normal s.p.c. manifold entering class (II). Then we know the following : 1°. The orbit space $\tilde{M} = M/U(1)$ becomes a differentiable manifold so that M is a differentiable $U(1)$-principal bundle over the base space $\tilde{M}$; 2°. The contact structure P defines a connection in the $U(1)$-principal bundle M and the basic form θ is the connection form. Let M' donote the $\mathbb{C}^*$-principal bundle over $\tilde{M}$ which is obtained from the $U(1)$-principal bundle M by enlarging the structure group $U(1)$ to $\mathbb{C}^* = GL(1, \mathbb{C})$.

Lemma 11.3 (cf. Hatakeyama [6]). (1) $\tilde{M}$ becomes a Kählerian manifold in a natural manner.

(2) The $\mathbb{C}^*$-principal bundle M' over $\tilde{M}$ becomes a holomorphic principal bundle in a natural manner and the s.p.c. structure S of M is induced from the injection $M \to M'$.

Proof. Let π be the projection $M \to \tilde{M}$. Then there are a unique almost complex structure $\tilde{I}$ and a unique Riemannian metric $\tilde{g}$

on $\tilde{M}$ such that $\pi_* IX = \tilde{I}\pi_* X$ and $g(X, Y) = \tilde{g}(\pi_* X, \pi_* Y)$ for $X, Y \in P_x$, $x \in M$. Let $\tilde{\omega}$ be the fundamental form associated with the hermitian structure $(\tilde{I}, \tilde{g})$. Then we have $\pi^*\tilde{\omega} = \omega = -d\theta$, whence $d\tilde{\omega} = 0$. This shows that $(\tilde{I}, \tilde{g})$ is a Kählerian structure, proving (1). As for assertion (2), we shall only explain how to define the almost complex structure I' on M'. Let Z^R(resp Z^I) be the vector field on M' induced from the 1-parameter group of right translations $M' \times \mathbb{R} \ni (x, t) \to x \cdot e^t \in M'$ (resp. $M' \times \mathbb{R} \ni (x, t) \to x \cdot e^{t\sqrt{-1}} \in M'$). Then there is a unique almost complex structure I' on M' such that I' is invariant under the right translations and such that $I'X = IX$ for $X \in P_x$, $x \in M$ and $I'Z^R_x = \xi_x = Z^I_x$ for $x \in M$.

Now let F be the holomorphic vector bundle over the Kählerian manifold $\tilde{M}$ associated with the holomorphic $\mathbb{C}^*$-bundle. Then we assert that the line bundle F is negative in the sense of Morrow - Kodaira [19]. Indeed, as we have already remarked, θ or preferably $\sqrt{-1}\theta$ difines a connection in the U(1)-bundle M over $\tilde{M}$. The curvature of this connection is $\sqrt{-1}d\theta = -\sqrt{-1}\,\omega = -\sqrt{-1}\pi^*\tilde{\omega}$. Hence the Chern class of F is represented by the 2-form $-\frac{\sqrt{-1}}{2\pi\sqrt{-1}}\tilde{\omega} = -\frac{1}{2\pi}\tilde{\omega}$, proving our assertion.

Conversely let F be any negative line bundle over a compact, complex manifold $\tilde{M}$, and M' the associated $\mathbb{C}^*$-principal bundle. Then we have a canonical U(1)-reduction M of M' such that M is s.p.c. in M'. (We can see this fact from Theorem 7.4 of [19].) Let ξ be the vector field on M induced from the 1-parameter group of right translations $M \times \mathbb{R} \ni (x, t) \to x\cdot e^{t\sqrt{-1}} \in M$. Then we find that ξ is an analytic basic field on M and hence that M is a normal s.p.c. manifold entering class (II).

11.4. The operator N. Let (M, ξ) be a compact, normal s.p.c. manifold. For every k, we define a differential operator

$$N : A^k(M) \longrightarrow A^k(M)$$

by

$$N\varphi = \sqrt{-1}L_\xi\varphi = \sqrt{-1}\,\nabla_\xi\varphi, \qquad \varphi \in A^k(M).$$

For any φ, $\psi \in A^k(M)$, we have

$$<N\varphi, \psi> = <\varphi, N\psi> + \sqrt{-1}\xi <\varphi, \psi> .$$

Therefore we see from Proposition 3.6 that the operator N is self-adjoint with respect to the inner product $(\ ,\ )$.

The operator N leaves invariant the subspaces $A^{p,q}(M)$ and $C^{p,q}(M)$, and commutes with the operators d, δ, d'', δ'', etc. It follows that the operator N operates on the cohomology groups, $H^{p,q}(M)$ and $H_*^{k-1,1}(M)$, as well as the spaces of harmonic forms, $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}_*^{k-1,1}(M)$.

The groups $H_{(\lambda)}^{p,q}(M)$ and the spaces $\mathcal{H}_{(\lambda)}^{p,q}(M)$. For each $\lambda \in \mathbb{R}$, we define a subspace $\mathcal{H}_{(\lambda)}^{p,q}(M)$ of $\mathcal{H}^{p,q}(M)$ by

$$\mathcal{H}_{(\lambda)}^{p,q}(M) = \{ \varphi \in \mathcal{H}^{p,q}(M) \mid N\varphi = \lambda\varphi \}.$$

Every $\varphi \in \mathcal{H}_{(\lambda)}^{p,q}(M)$ satisfies the differential equation

$$(\Delta'' + N^2)\varphi = \lambda^2\varphi .$$

Since the operator $\Delta'' + N^2$ is a self-adjoint, strongly elliptic differential operator, we see that $\mathcal{H}_{(\lambda)}^{p,q}(M)$ is finite dimensional (for any (p, q) and any λ) and that the eigenvalues of the operator $N : \mathcal{H}^{p,q}(M) \longrightarrow \mathcal{H}^{p,q}(M)$ form a discrete subset (without accumulating point) of $\mathbb{R}$. Since $\mathcal{H}^{p,q}(M)$ is finite dimensional if $q \neq 0$, $n - 1$, we have

$$\mathcal{H}^{p,q}(M) = \oplus \sum_{\lambda} \mathcal{H}^{p,q}_{(\lambda)}(M) \quad \text{if} \quad q \neq 0,\ n - 1.$$

Since $N\#_A = -\#_A N$ and $\#_A \mathcal{H}^{p,q}(M) = \mathcal{H}^{n-p,n-q-1}(M)$, we have

$$\#_A \mathcal{H}^{p,q}_{(\lambda)}(M) = \mathcal{H}^{n-p,n-q-1}_{(-\lambda)}(M).$$

Now, for each $\lambda \in \mathbb{R}$, we define a subgroup $H^{p,q}_{(\lambda)}(M)$ of $H^{p,q}(M)$ by

$$H^{p,q}_{(\lambda)}(M) = \{ c \in H^{p,q}(M) \mid Nc = \lambda c \}.$$

Clearly we have

$$H^{p,q}_{(\lambda)}(M) \cong \mathcal{H}^{p,q}_{(\lambda)}(M).$$

The groups $H^{k-1,1}_{*,(\lambda)}(M)$ and the spaces $\mathcal{H}^{k-1,1}_{*,(\lambda)}(M)$. In the same way as above, we define the subgroups $H^{k-1,1}_{*,(\lambda)}(M) \subset H^{k-1.1}_{*}(M)$ and the subspaces $\mathcal{H}^{k-1,1}_{*,(\lambda)}(M) \subset \mathcal{H}^{k-1,1}_{*}(M)$. Then we have

$$\mathcal{H}^{k-1,1}_{*}(M) = \oplus \sum_{\lambda} \mathcal{H}^{k-1,1}_{*,(\lambda)}(M),$$

$$H^{k-1,1}_{*,(\lambda)}(M) \cong \mathcal{H}^{k-1,1}_{*,(\lambda)}(M).$$

Note that if the normal s.p.c. manifold M enters class (I), then the eigenvalues of the operators $N : \mathcal{H}^{p,q}(M) \longrightarrow \mathcal{H}^{p,q}(M)$ and N :

$H_*^{k-1,1}(M) \longrightarrow H_*^{k-1,1}(M)$ are all integers.

In §13, we shall make a detailed study of the groups $H_{(\lambda)}^{p,q}(M)$, $H_{*,(\lambda)}^{k-1,1}(M)$ and $H_0^k(M)$.

§12. The double complex $\{B^{p,q}(M), \partial, \bar{\partial}\}$

In this section and the subsequent section, (M, ξ) will be a compact, normal s.p.c. manifold of dimension $2n - 1 \geq 5$.

12.1. The fundamental operators. We put as follows :

$$B^k(M) = \Lambda^k(\mathbb{C}P)^* \subset A^k(M),$$

$$B^{p,q}(M) = \Lambda^p S^* \otimes \Lambda^q \bar{S}^* = B^{p+q}(M) \cap C^{p,q}(M).$$

Then $B^k(M)$ is equipped with the inner product $< , >$ as a subbundle of $A^k(M)$, and

$$B^k(M) = \oplus \sum_{p+q=k} B^{p,q}(M).$$

The operator $*_B$. For each k, there is a unique operator

$$*_B : B^k(M) \to B^{2n-2-k}(M)$$

having the following properties :

1) $*_B$ is a real operator, i.e., $\overline{*_B \varphi} = *_B \overline{\varphi}$;

2) $<\varphi, \psi>(d\theta)^{n-1} = (n-1)!\, \varphi \wedge \overline{*_B \psi}$;

3) $*_B *_B \varphi = (-1)^k \varphi, \quad \varphi \in B^k(M).$

The operators L and Λ. By using the cross section $\omega = - d\theta$ of $B^{1,1}(M)$, we define an operator

$$L : B^k(M) \longrightarrow B^{k+2}(M)$$

by
$$L\varphi = \omega \wedge \varphi, \qquad \varphi \in B^k(M).$$

Let Λ be the adjoint operator of L with respect to the inner product $<\ ,\ >$. We have

$$\Lambda\varphi = (-1)^k *_B L *_B \varphi, \qquad \varphi \in B^k(M).$$

Lemma 12.1.

$$\Lambda L\varphi = L\Lambda\varphi + (n-k-1)\varphi, \qquad \varphi \in B^k(M).$$

We put as follows :

$$\mathcal{B}^k(M) = \Gamma(B^k(M)),$$

$$\mathcal{B}^{p,q}(M) = \Gamma(B^{p,q}(M)).$$

Then $\mathcal{B}^k(M)$ is equipped with the inner product $(\ ,\ )$ as a subspace of $\mathcal{A}^k(M)$.

The operator N. Consider the self-adjoint operator $N : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$, which leaves invariant the subspace $\mathcal{B}^k(M)$. The operator $N : \mathcal{B}^k(M) \longrightarrow \mathcal{B}^k(M)$ leaves invariant the subspace $\mathcal{B}^{p,q}(M)$, and commutes with the operators $*_B$, L and Λ.

The operators $\partial, \bar{\partial}, \vartheta$ and $\bar{\vartheta}$. Let $\varphi \in \mathcal{B}^{p,q}(M)$. As is easily observed, the exterior derivative $d\varphi$ can be written uniquely in the form :

$$d\varphi \equiv \partial\varphi + \bar{\partial}\varphi \quad (\mathrm{mod}\ \theta),$$

where $\partial\varphi \in \mathcal{B}^{p+1,q}(M)$ and $\bar{\partial}\varphi \in \mathcal{B}^{p,q+1}(M)$. In this way we get operators

$$\partial,\ \bar{\partial} \ : \mathcal{B}^{k}(M) \to \mathcal{B}^{k+1}(M)$$

with $\partial\mathcal{B}^{p,q}(M) \subset \mathcal{B}^{p+1,q}(M)$ and $\bar{\partial}\mathcal{B}^{p,q}(M) \subset \mathcal{B}^{p,q+1}(M)$. Clearly we have have $\bar{\partial}\varphi = \overline{\partial\bar{\varphi}}$. Note that the operators ∂ and $\bar{\partial}$ depend on the choice of the analytic basic field ξ.

Let ϑ and $\bar{\vartheta}$ be the adjoint operators of $\bar{\partial}$ and ∂ respectively. We have $\bar{\vartheta}\varphi = \overline{\vartheta\bar{\varphi}}$ and $\vartheta = -*_B\partial*_B$.

In terms of the canonical affine connection ∇, $\bar{\partial}\varphi$ and $\vartheta\varphi$, $\varphi \in \mathcal{B}^{p,q}(M)$, may be described as follows:

$$(\bar{\partial}\varphi)(X_1,\dots, X_p, \bar{Y}_1,\dots, \bar{Y}_{q+1})$$
$$= (-1)^p \sum_j (-1)^{j+1}(\nabla_{\bar{Y}_j}\varphi)(X_1,\dots, X_p, \bar{Y}_1,\dots, \hat{\bar{Y}}_j,\dots, \bar{Y}_{q+1}),$$
$$(\vartheta\varphi)(X_1,\dots, X_p, \bar{Y}_1,\dots, \bar{Y}_{q-1})$$
$$= (-1)^{p+1} \sum_k (\nabla_{e_k}\varphi)(X_1,\dots, X_p, \bar{e}_k, \bar{Y}_1,\dots, \bar{Y}_{q-1}),$$

where $X_1,\dots, X_p, Y_1,\dots, Y_{q+1} \in S_x$.

Lemma 12.2. (1) $\partial^2 = \bar{\partial}^2 = 0$ and

$$\partial\bar{\partial} + \bar{\partial}\partial = -\sqrt{-1}\, LN.$$

(2) $\vartheta^2 = \bar{\vartheta}^2 = 0$ and

$$\vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = \sqrt{-1}\, \Lambda N.$$

Proof. Let $\varphi \in \mathcal{B}^{p,q}(M)$. Then there is a unique $\alpha \in \mathcal{B}^{p+q}(M)$ such that

$$d\varphi = (\partial + \bar{\partial})\varphi + \theta \wedge \alpha .$$

Since $L_\xi \varphi = \xi \rfloor d\varphi$, it follows that $\alpha = L_\xi \varphi = -\sqrt{-1}\, N\varphi \in \mathcal{B}^{p,q}(M)$. Hence

$$(12.1) \quad d\varphi = (\partial + \bar{\partial})\varphi - \sqrt{-1}\, \theta \wedge N\varphi .$$

Since $d\theta = -\omega$, we find

$$(\partial + \bar{\partial})^2\varphi \equiv -\sqrt{-1}\, \omega \wedge N\varphi \pmod{\theta}.$$

This clearly means

$$\partial^2\varphi + \bar{\partial}^2\varphi + (\partial\bar{\partial} + \bar{\partial}\partial)\varphi = -\sqrt{-1}\, LN\varphi,$$

from which follows immediately (1). (2) is is easy from (1).

The first assertion of Lemma 12.2 indicates that the collection $\{ \mathcal{B}^{p,q}(M), \partial, \bar{\partial} \}$ gives a double complex (in a generalized sense) where the relation $\partial\bar{\partial} + \bar{\partial}\partial = 0$ is not satisfied.

Lemma 12.3.

(1) $\bar{\partial}\Lambda - \Lambda\bar{\partial} = -\sqrt{-1}\,\bar{\vartheta}, \quad \partial\Lambda - \Lambda\partial = \sqrt{-1}\,\vartheta.$

(2) $\vartheta L - L\vartheta = -\sqrt{-1}\,\partial, \quad \bar{\vartheta}L - L\bar{\vartheta} = \sqrt{-1}\,\bar{\partial}.$

(3) $\bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial} = 0, \quad \partial \; + \; \partial = 0.$

The proof of this fact is just analogous to the case of a Kählerian manifold (e.g., see [19]). We also note that L (resp. Λ) commutes with ∂ and $\bar{\partial}$ (resp. with ϑ and $\bar{\vartheta}$).

The operators $\square$ and $\bar{\square}$. Let $\square$ (resp. $\bar{\square}$) denote the operator $\vartheta\bar{\partial} + \bar{\partial}\vartheta$ (resp. $\bar{\vartheta}\partial + \partial\bar{\vartheta}$).

Lemma 12.4.

$$\bar{\square}\varphi + (n - k - 1)N\varphi = \square\varphi, \quad \varphi \in B^k(M).$$

Proof. By (1) of Lemma 12.3, we have

$$(\partial\bar{\partial} + \bar{\partial}\partial)\Lambda - \Lambda(\partial\bar{\partial} + \bar{\partial}\partial) = \sqrt{-1}(\square - \bar{\square}).$$

Therefore it follows from Lemmas 12.1 and 12.2 that

$$(\square - \bar{\square})\varphi = (\Lambda L - L\Lambda)N\varphi = (n - k - 1)N\varphi.$$

12.2. The complex $\{ \mathcal{B}^{p,q}(M), \bar{\partial} \}$. In this paragraph we shall observe the complex $\{ \mathcal{B}^{p,q}(M), \bar{\partial} \}$. The cohomology groups of this complex will be denoted by $\tilde{H}^{p,q}(M)$.

For any $X, Y, Z \in \Gamma(S)$, we have

$$\nabla_{\bar{X}}(fZ) = f\nabla_{\bar{X}}Z + \bar{X}f \cdot Z \quad (f \in \mathbb{C}\mathcal{F}(M)),$$

$$\nabla_{\bar{X}}(\nabla_{\bar{Y}}Z) - \nabla_{\bar{Y}}(\nabla_{\bar{X}}Z) - \nabla_{[\bar{X},\bar{Y}]}Z = R(\bar{X}, \bar{Y}) = 0.$$

This fact means that S becomes a holomorphic vector bundle over the s.p.c. manifold M with respect to the operator $\bar{\partial}_S : \Gamma(S) \longrightarrow \Gamma(S \otimes \bar{S}^*)$ defined by

$$(\bar{\partial}_S Z)(\bar{X}) = \nabla_{\bar{X}}Z, \qquad X, Z \in \Gamma(S).$$

Remark. Let ξ' be the image of the analytic basic field ξ by the natural map $T(M) \rightarrow \hat{T}(M)$ $(= \mathbb{C}T(M)/\bar{S})$, being a holomorphic cross section of $\hat{T}(M)$. Let $\mathbb{C}\xi'$ denote the 1-dimensional subbundle of $\hat{T}(M)$ spanned by ξ'. Then the composition of the natural maps $S \rightarrow \hat{T}(M)$ and $\hat{T}(M) \rightarrow \hat{T}(M)/\mathbb{C}\xi'$ gives an isomorphism of S onto $\hat{T}(M)/\mathbb{C}\xi'$ as holomorphic vector bundles.

S being a holomorphic vector bundle, so is $F^p = \Lambda^p S^*$. Just as in the case of the complex $\{\mathcal{C}^{p,q}(M), d''\}$, the space $\mathcal{B}^{p,q}(M)$ may be identified with the space $\mathcal{C}^{q}(M, F^{p})$, and $\bar{\partial} = (-1)^{p}\bar{\partial}_{F^p}$. Therefore from the general harmonic theory developed in §6, we know the following (cf. 7.2) : 1°. $\mathcal{H}^{p,q}(M) = \{ \varphi \in \mathcal{B}^{p,q}(M) \mid \Box\varphi = 0 \}$ is finite dimensional for any (p, q) with $q \neq 0, n-1$; 2°. $\tilde{H}^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$ for any (p, q).

We define operators

$$R,\ R_*^{(1)},\ R_*^{(2)} : B^{p,q}(M) \to B^{p,q}(M)$$

respectively as follows :

$$(R\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= \sum_{i,j,k} (-1)^{j} \varphi(X_1, \ldots, \underset{i^{\text{th}} \text{ place}}{R(e_k, \bar{Y}_j)X_i}, \ldots, X_p, \bar{e}_k, \bar{Y}_1, \ldots, \hat{\bar{Y}}_j, \ldots, \bar{Y}_q),$$

$$(R_*^{(1)}\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= \sum_{j} \varphi(X_1, \ldots, X_p, \bar{Y}_1, \ldots, R_*\bar{Y}_j, \ldots, \bar{Y}_q),$$

$$(R_*^{(2)}\varphi)(X_1, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q)$$

$$= \sum_{i} \varphi(X_1, \ldots, R_*X_i, \ldots, X_p, \bar{Y}_1, \ldots, \bar{Y}_q),$$

where $\varphi \in B^{p,q}(M)$ and $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in S_x$. Note that these operators are all self-adjoint with respect to the inner product $\langle \, , \, \rangle$. Using these operators, we further define self-adjoint operators

$$Q_1^{p,q}, \quad Q_2^{p,q}, \quad Q^{p,q} \ : \ B^{p,q}(M) \to B^{p,q}(M)$$

respectively by

$$Q_1^{p,q} = R + R_*^{(1)},$$

$$Q_2^{p,q} = R + R_*^{(2)},$$

$$Q^{p,q} = R + \frac{n-q-1}{n-1} R_*^{(1)} + \frac{q}{n-1} R_*^{(2)} .$$

Let us now define semi-norms $\| \ \|_{\bar{S}}$ and $\| \ \|_S$ in $B^{p,q}(M)$ respectively by

$$\| \varphi \|_{\bar{S}}^2 = \int (\sum_i \langle \nabla_{\bar{e}_i} \varphi , \nabla_{\bar{e}_i} \varphi \rangle) \, dv,$$

$$\| \varphi \|_S^2 = \int (\sum_i \langle \nabla_{e_i} \varphi , \nabla_{e_i} \varphi \rangle) \, dv.$$

Then we have the following

Proposition 12.5 (cf. Theorem 5.2). For any $\varphi \in B^{p,q}(M)$, we have the equalities :

(1) $(\Box \varphi, \varphi) = \| \varphi \|_{\bar{S}}^2 - q(N\varphi, \varphi) + (Q_1^{p,q}\varphi, \varphi)$.

(2) $(\square\varphi, \varphi) = \|\varphi\|_S^2 + (n-q-1)(N\varphi, \varphi) + (Q_2^{p,q}\varphi, \varphi)$.

(3) $(\square\varphi, \varphi) = \frac{n-q-1}{n-1}\|\varphi\|_{\bar{S}}^2 + \frac{q}{n-1}\|\varphi\|_S^2 + (Q^{p,q}\varphi, \varphi)$.

12.3. The groups $\tilde{H}^{p,q}_{(\lambda)}(M)$ and the spaces $\tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M)$. The self-adjoint operator $N : \mathcal{B}^k(M) \to \mathcal{B}^k(M)$ leaves invariant the subspaces $\mathcal{B}^{p,q}(M)$ and commutes with the operators $\bar{\partial}, \vartheta$, etc. Hence it operates on the cohomology groups $\tilde{H}^{p,q}(M)$ as well as the spaces of harmonic forms $\tilde{\mathcal{H}}^{p,q}(M)$. In the same manner as in 11.4, we define the subgroups $\tilde{H}^{p,q}_{(\lambda)}(M) \subset \tilde{H}^{p,q}(M)$ and the subspaces $\tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M) \subset \tilde{\mathcal{H}}^{p,q}(M)$. Then $\tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M)$ is finite dimensional (for any (p, q) and any λ), and we have

$$\tilde{\mathcal{H}}^{p,q}(M) = \oplus \sum_{\lambda} \tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M) \quad \text{if} \quad q \neq 0, n-1,$$

$$\tilde{H}^{p,q}_{(\lambda)}(M) \cong \tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M).$$

Let $\#_B$ denote the operator $\mathcal{B}^k(M) \ni \varphi \to \overline{*_B\varphi} \in \mathcal{B}^{2n-k-2}(M)$. Then we have $N\#_B = -\#_B N$ and $\#_B\tilde{\mathcal{H}}^{p,q}(M) = \tilde{\mathcal{H}}^{n-p-1,n-q-1}(M)$. Hence

$$\#_B\tilde{\mathcal{H}}^{p,q}_{(\lambda)}(M) = \tilde{\mathcal{H}}^{n-p-1,n-q-1}_{(-\lambda)}(M).$$

We also note that the eigenvalues of the operator $N : \tilde{\mathcal{H}}^{p,q}(M) \longrightarrow \tilde{\mathcal{H}}^{p,q}(M)$ are all integers, provided M enters class (I).

By Lemma 12.4, we have

Proposition 12.6. If $\lambda > 0$ and $p+q < n-1$, then $\tilde{H}^{p,q}_{(\lambda)}(M) = 0$.

On account of Proposition 12.7 below, we know that Proposition 12.6 generalizes Nakano's vanishing theorem concerning negative line bundles over compact complex manifolds (e.g. [19]).

For the rest of this paragraph we assume M to enter class (II). Consider the holomorphic $\mathbb{C}^*$-principal bundle M' over the compact complex manifold $\tilde{M} = M/U(1)$ and the associated line bundle F over $\tilde{M}$ (see 11.3). For any integers p and m, denote by $\Omega^p(F^m)$ the sheaf of local holomorphic p-forms with values in the line bundle F^m, the m-th power of F. That is, $\Omega^p(F^m)$ is the sheaf of local holomorphic cross sections of the holomorphic vector bundle $F^{m,p} = F^m \otimes \Lambda^p(\hat{T}(\tilde{M}))^*$. As is well known, the q-th cohomology group $H^q(\tilde{M}, \Omega^p(F^m))$ of the sheaf $\Omega^p(F^m)$ is isomorphic with the cohomology group $H^q(\tilde{M}, F^{m,p})$ (the Dolbeault isomorphism).

Proposition 12.7. For any integers p, q and m, we have the isomorphism :

$$\tilde{H}^{p,q}_{(m)}(M) \cong H^q(\tilde{M}, \Omega^p(F^m)).$$

Proof. Put

$$\mathcal{B}^{p,q}_{(m)}(M) = \{ \varphi \in \mathcal{B}^{p,q}(M) \mid N\varphi = m\varphi \}.$$

Then $\varphi \in \mathcal{B}^{p,q}(M)$ is in $\mathcal{B}^{p,q}_{(m)}(M)$ if and only if

$$R_a^* \varphi = a^{-m}\varphi, \qquad a \in U(1),$$

where R_a denotes the right translation $M \ni x \longrightarrow x \cdot a \in M$. To prove Proposition 12.7, it is sufficient to show that there are (natural) isomorphisms $\mathcal{B}^{p,q}_{(m)}(M) \ni \varphi \to \tilde{\varphi} \in C^q(\tilde{M}, F^{m,p})$ such that $\widetilde{(\bar{\partial}\varphi)} = (-1)^p\, \bar{\partial}_{F^{m,p}} \tilde{\varphi}$, $\varphi \in \mathcal{B}^{p,q}_{(m)}(M)$.

Let π' (resp. π) be the projection of M' (resp. of M) onto $\tilde{M}$. M' being a holomorphic principal bundle, we have an open covering $\{U_\alpha\}$ of $\tilde{M}$ and, for each α, a holomorphic trivialization ϕ : $\pi'^{-1}(U_\alpha) \ni z \longrightarrow (\pi'(z), f_\alpha(z)) \in U_\alpha \times \mathbb{C}^*$. Let $\{g_{\alpha\beta}\}$ be the system of (homolorphic) transition functions associated with the trivializations ϕ_α. Then we have

$$f_\alpha(za) = f_\alpha(z)a,$$

$$f_\alpha(z) = g_{\alpha\beta}(\pi'(z))\, f_\beta(z), \qquad z \in \pi'^{-1}(U_\alpha), \qquad a \in \mathbb{C}^*.$$

Now let u_α be the restriction of f_α to $\pi^{-1}(U_\alpha)$. Then u_α is a holomorphic function on the s.p.c. manifold M : $d''u_\alpha = \bar\partial u_\alpha = 0$. Furthermore from the equalities above for f_α, we obtain

$$R_a^* u_\alpha = u_\alpha \cdot a, \quad a \in U(1),$$

$$u_\alpha / u_\beta = \pi^* g_{\alpha\beta} .$$

Take any $\varphi \in B^{p,q}_{(m)}(M)$ and put $\varphi_\alpha = u_\alpha^m \cdot \varphi$. Then we have, for any $a \in U(1)$,

$$R_a^* \varphi_\alpha = (R_a^* u_\alpha)^m \cdot R_a^* \varphi = u_\alpha^m \cdot a^m \cdot a^{-m} \cdot \varphi = \varphi_\alpha$$

and $\xi \rfloor \varphi_\alpha = 0$. It follows that there is a unique $\tilde\varphi_\alpha \in C^{p,q}(U_\alpha)$ such that $\varphi_\alpha = \pi^* \tilde\varphi_\alpha$. Let α and β be such that $U_\alpha \cap U_\beta \neq \phi$. Then we have

$$\varphi_\alpha = (u_\alpha/u_\beta)^m \varphi_\beta = (\pi^* g_{\alpha\beta})^m \varphi_\beta , \quad \text{whence } \tilde\varphi_\alpha = (g_{\alpha\beta})^m \tilde\varphi_\beta .$$

This means that the collection $\{\tilde\varphi_\alpha\}$ defines an element, say $\tilde\varphi$, of $C^q(\tilde M, F^{m,p})$. It is easy to see that the assignment $\varphi \longrightarrow \tilde\varphi$ gives an isomorphism of $B^{p,q}_{(m)}(M)$ onto $C^q(\tilde M, F^{m,p})$. Let $\varphi \in B^{p,q}_{(m)}(M)$ Since $\bar\partial \varphi_\alpha = \pi^* d'' \tilde\varphi_\alpha$ and $\bar\partial u_\alpha = 0$, we see that

$$\bar\partial \varphi = \bar\partial (u_\alpha^{-m} \cdot \varphi_\alpha) = \bar\partial (u_\alpha^{-m}) \wedge \varphi_\alpha + u_\alpha^{-m} \bar\partial \varphi_\alpha$$

$$= u_\alpha^{-m}\, \pi^* d''\, \tilde{\varphi}_\alpha ,$$

whence $(\widetilde{\bar\partial\varphi})_\alpha = d''\,\tilde{\varphi}_\alpha$. Since $(-1)^p\,\bar\partial_{F^{m,p}}\,\tilde{\varphi}$ is defined by the collection $\{ d''\,\tilde{\varphi}_\alpha \}$, we get $(\widetilde{\bar\partial\varphi}) = (-1)^p\,\bar\partial_{F^{m,p}}\tilde{\varphi}$. We have thereby proved Proposition 12.7.

12.4. Normal s.p.c. manifolds with vanishing curvature. Let n be an integer ≥ 3. Let $\mathfrak{g} = \sum_{i=1}^{\infty} \mathfrak{g}_i$ be a graded Lie algebra over $\mathbb{R}$ and let I be a complex structure on the vector space $\mathfrak{g}_1$. For the pair $(\mathfrak{g}, I)$ assume the following:

1) $\mathfrak{g}_i = 0$ if $i \geq 3$, $\dim \mathfrak{g}_2 = 1$ and $\dim_{\mathbb{R}} \mathfrak{g}_1 = 2(n-1)$;

2) $[IX, IY] = [X, Y]$, $X, Y \in \mathfrak{g}_1$;

3) The hermitian quadratic form $\mathfrak{g}_1 \ni X \longrightarrow [IX, X] \in \mathfrak{g}_2$ $(\cong \mathbb{R})$ is definite.

Let G be the simply connected Lie group whose Lie algebra (of left invariant vector fields) is given by $\mathfrak{g}$. Consider the exponential map $\exp : \mathfrak{g} \longrightarrow G$. Then exp is a diffeomorphism of $\mathfrak{g}$ onto G, $G_2 = \exp \mathfrak{g}_2$ is the center of G, and

$$\exp X \exp Y = \exp (X + Y) \exp(\tfrac{1}{2} [X, Y]), \quad X, Y \in \mathfrak{g}_1 .$$

We define a subspace $\mathfrak{s}$ of $\mathbb{C}\mathfrak{g}_1$ by

$$\mathfrak{s} = \{X - \sqrt{-1}\, IX \mid X \in \mathfrak{g}_1 \} .$$

Then $\mathfrak{s}$ induces a left invariant subbundle S of $\mathbb{C}T(M)$, and, as is easily observed, S gives a s.p.c. structure on G. The manifold G together with the structure S is called the standard s.p.c. manifold (cf. [30]). We note that the s.p.c. manifold G can be realized as the real hypersurface of $\mathbb{C}^n$ defined by

$$\operatorname{Im} Z_n = \frac{1}{2} \sum_{i=1}^{n-1} |Z_i|^2 .$$

We fix a base ξ of $\mathfrak{g}_2$ such that the quadratic form $\mathfrak{g}_1 \ni X \to [IX, X] \in \mathfrak{g}_2$ is positive definite, where $\mathfrak{g}_2$ should be identified with $\mathbb{R}$ w.r.t. ξ. It is easy to see that ξ is an analytic basic field on G and that the canonical affine connection ∇ of (G, ξ) is uniquely determined by the property : Every left invariant vector field X is parallel w.r.t. ∇, i.e., $\nabla X = 0$. It follows that the curvature of ∇ vanishes.

Let Γ be a discrete subgroup of G such that the space $M =$

$\Gamma\backslash G$ of right cosets of G by Γ is compact. Then every left invariant vector field X is projectable on M, i.e., there is a unique vector field X' on M such that X and X' are π-related, π being the projection $G \to M$. It follows that the s.p.c. structure S on G induces a s.p.c. structure S' on M in a natural manner and that ξ' is an analytic basic field on the s.p.c. manifold M.

Proposition 12.8.

(1) $\tilde{H}^{p,q}_{(\lambda)}(M) = 0$ if $\lambda \neq 0$ and $1 \leq q \leq n-2$.

(2) $\tilde{H}^{p,q}_{(0)}(M) \cong \Lambda^p \mathfrak{s}^* \otimes \Lambda^q \bar{\mathfrak{s}}^*$ for all (p, q).

Proof. Let $\varphi \in B^{p,q}(M)$. Since the curvature of (M, ξ') vanishes, we see from Proposition 12.5 that $\varphi \in \tilde{H}^{p,q}_{(\lambda)}(M)$ if and only if

$$N\varphi = \lambda\varphi,$$

$$\|\varphi\|^2_{\bar{S}} - q\lambda\|\varphi\|^2 = 0,$$

$$\|\varphi\|^2_S + (n-q-1)\lambda\|\varphi\|^2 = 0.$$

Hence $\tilde{H}^{p,q}_{(\lambda)}(M) = 0$ if $\lambda \neq 0$ and $1 \leq q \leq n-2$, and φ is in $\tilde{H}^{p,q}_{(0)}(M)$ if and only if $\nabla'\varphi = 0$ (or $\nabla(\pi^*\varphi) = 0$), where ∇' is the canonical affine connection of (M, ξ'). Since $\nabla_X Y = 0$ for all $X, Y \in \mathfrak{g}$, this last condition is equivalent to the condition that $\pi^*\varphi$ is left invariant. Hence $\tilde{H}^{p,q}_{(0)}(M) \cong \Lambda^p\mathfrak{s}^* \otimes \Lambda^q\bar{\mathfrak{s}}^*$, completing the proof of Proposition 12.8.

Let $e_1, \ldots, e_{2n-2}$ be a base of $\mathfrak{g}_1$ over $\mathbb{R}$ such that $[e_i, e_j] = a_{ij}\xi$ with some integers a_{ij}, and let

$$\gamma = \mathbf{Z}\cdot\xi + \sum_i \mathbf{Z}\cdot e_i .$$

Then we see that $\Gamma = \exp\gamma$ is a discrete subgroup of G and that $M = \Gamma\backslash G$ is compact. Furthermore we see that the normal s.p.c. manifold (M, ξ) enters class (II) and that the complex manifold $M/U(1) = \Gamma\backslash G/G_2$ is holomorphically isomorphic with the abelian variety $\mathfrak{g}_1/\sum_i \mathbf{Z}\cdot e_i$.

§13. Reduction theorems for the cohomology groups $H^{p,q}_{(\lambda)}(M)$ and $H^k_0(M)$

In this section we shall describe the groups $H^{p,q}_{(\lambda)}(M)$, $H^{k-1,1}_{*,\ (\lambda)}(M)$ and $H^k_0(M)$ in terms of the group $\tilde{H}^{p,q}_{(\lambda)}(M)$.

13.1. The groups $H^{p,q}_{(0)}(M)$. Using the basic form θ, we define a map

$$e(\theta) : \mathcal{B}^{k-1}(M) \longrightarrow A^k(M)$$

by

$$e(\theta)\varphi = \theta \wedge \varphi, \qquad \varphi \in \mathcal{B}^{k-1}(M).$$

Then we have

$$A^k(M) = \mathcal{B}^k(M) \oplus e(\theta)\mathcal{B}^{k-1}(M),$$

$$C^{p,q}(M) = \mathcal{B}^{p,q}(M) \oplus e(\theta)\ \mathcal{B}^{p-1,q}(M).$$

Furthermore the map $e(\theta)$ preserves the inner product $(\ ,\ )$, and

$$\#_A\varphi = (-1)^{k-1}e(\theta)\#_B\varphi, \qquad \varphi \in \mathcal{B}^{k-1}(M).$$

We denote by $\mathcal{H}^k(M)$ the space of harmonic k-forms associated with the Riemannian metric h. Let d* be the adjoint operator of the exterior differentiation d. By definition, $\varphi \in A^k(M)$ is in

$\mathcal{H}^k(M)$ if and only if φ satisfies the equations: $d\varphi = d^*\varphi = 0$.

We have

$$\#_A \mathcal{H}^k(M) = \mathcal{H}^{2n-k-1}(M).$$

We also define subspaces $K^{p,q}_\Lambda(M)$ and $K^{p,q}_L(M)$ of $\tilde{\mathcal{H}}^{p,q}_{(0)}(M)$ by

$$K^{p,q}_\Lambda(M) = \{ \varphi \in \tilde{\mathcal{H}}^{p,q}_{(0)}(M) \mid \Lambda\varphi = 0 \},$$

$$K^{p,q}_L(M) = \{ \varphi \in \tilde{\mathcal{H}}^{p,q}_{(0)}(M) \mid L\varphi = 0 \}.$$

Since $\Lambda = (-1)^k \#_B L \#_B \varphi$, $\varphi \in \mathcal{B}^k(M)$, and $\#_B \tilde{\mathcal{H}}^{p,q}_{(0)}(M) = \tilde{\mathcal{H}}^{n-p-1,n-q-1}_{(0)}(M)$, we have

$$K^{p,q}_L(M) = \#_B K^{n-p-1,n-q-1}_\Lambda(M),$$

and by Lemma 12.1,

$$K^{p,q}_L(M) = 0 \quad \text{if} \quad p + q \leq n - 2,$$

$$K^{p,q}_\Lambda(M) = 0 \quad \text{if} \quad p + q \geq n,$$

$$K^{p,q}_L(M) = K^{p,q}_\Lambda(M) \quad \text{if} \quad p + q = n - 1.$$

These being prepared, we state the next two theorems.

Theorem 13.1.

(1) $\mathcal{H}^k(M) = \oplus \sum_{p+q=k} K^{p,q}_\Lambda(M)$ if $k \leq n-1$.

(2) $\mathcal{H}^k(M) = \oplus \sum_{p+q=k} e(\theta)\, \mathcal{K}_L^{p-1,q}(M)$ if $k \geq n$.

Theorem 13.2.

(1) $\mathcal{H}_{(0)}^{p,q}(M) = \mathcal{K}_\Lambda^{p,q}(M)$ if $p + q \leq n-1$.

(2) $\mathcal{H}_{(0)}^{p,q}(M) = e(\theta)\mathcal{K}_L^{p-1,q}(M)$ if $p + q \geq n$.

From these theorems it follows immediately that

$$\mathcal{H}^k(M) = \oplus \sum_{p+q=k} \mathcal{H}_{(0)}^{p,q}(M).$$

Consequently we get

Corollary (cf. Naruki [23]). For every k we have the isomorphism :

$$H^k(M) \cong \sum_{p+q=k} H_{(0)}^{p,q}(M).$$

Proof of Theorem 13.1. Every $\alpha \in A^k(M)$ can be written uniquely in the form : $\alpha = \pi_0\alpha + e(\theta)\pi_1\alpha$,

where $\pi_0\alpha \in \mathcal{B}^k(M)$ and $\pi_1\alpha \in \mathcal{B}^{k-1}(M)$.

Lemma 13.3. Let $\alpha \in A^k(M)$.

(1) $\pi_0 d\alpha = (\partial + \bar{\partial})\pi_0\alpha - L\pi_1\alpha$,

$\pi_1 d\alpha = -(\sqrt{-1}\, N\pi_0\alpha + (\partial + \bar{\partial})\pi_1\alpha)$.

(2) $\pi_0 d^*\alpha = (\vartheta + \bar{\vartheta})\pi_0\alpha + \sqrt{-1}\, N\pi_1\alpha$,

$$\pi_1 d^*\alpha = -(\Lambda\pi_0\alpha + (\vartheta + \bar{\vartheta})\pi_1\alpha).$$

Proof. Using (12.1), we obtain

$$d\alpha = d\pi_0\alpha - \omega \wedge \pi_1 \quad - \theta \wedge d\pi_1\alpha$$

$$= (\partial + \bar{\partial})\pi_0\alpha - L\pi_1\alpha - \theta \wedge (\sqrt{-1}\, N\pi_0\alpha + (\partial + \bar{\partial})\pi_1\alpha).$$

Hence we get (1). (2) is easily obtained from (1).

Let $\alpha = \varphi + e(\theta)\psi \in A^k(M)$, where $\varphi = \pi_0\alpha$ and $\psi = \pi_1\alpha$.
By Lemma 13.3, α is in $\mathcal{H}^k(M)$ if and only if φ and ψ satisfy the equations :

$$(13.\ 1) \quad \begin{cases} (\partial + \bar{\partial})\varphi = L\psi , \\ (\partial + \bar{\partial})\psi = -\sqrt{-1}N\varphi , \\ (\vartheta + \bar{\vartheta})\varphi = -\sqrt{-1}N\psi , \\ (\vartheta + \bar{\vartheta})\psi = -\Lambda\varphi . \end{cases}$$

To prove Theorem 13.1, it is sufficent to verify the first assertion (1), because the second assertion (2) can be obtained from the first by utilizing the dualities given by $\#_A$ and $\#_B$.
Putting $\mathcal{K}_\Lambda^k(M) = \sum_{p+q=k} \mathcal{K}_\Lambda^{p,q}(M)$, we first show that $\mathcal{K}_\Lambda^k(M) \subset \mathcal{H}^k(M)$.

Let $\varphi \in K_\Lambda^k(M)$. Then $\bar\partial\varphi = \vartheta\varphi = N\varphi = \Lambda\varphi = 0$. By Lemma 12.4 we have $\bar\Box\varphi = \Box\varphi = 0$, whence $\partial\varphi = \bar\vartheta\varphi = 0$. Therefore φ satisfies equations (13. 1) , i.e., $\varphi \in H^k(M)$.

Conversely we have $H^k(M) \subset K_\Lambda^k(M)$ by the following

Lemma 13. 4. Let $\alpha = \varphi + e(\theta)\psi \in H^k(M)$.

(1) $N\varphi = N\psi = 0$.

(2) $2\,\Box\varphi + L\Lambda\varphi = 2\sqrt{-1}\,\bar\partial\psi = -2\sqrt{-1}\,\partial\psi$.

(3) $\bar\partial\partial\varphi = 0$.

(4) $\Lambda\varphi = 0$.

(5) $\psi = 0$, and $\alpha = \varphi \in K_\Lambda^k(M)$.

Proof. In the proof below, we shall freely use conditon (13. 1) and Lemmas 12.1 $\sim$ 12.4 without comments.

(1) Since $L_\xi\theta = L_\xi g = 0$, we have $L_\xi h = L_\xi(g + \theta^2) = 0$. α being a harmonic form for the Riemannian metric h, it follows that $L_\xi\alpha = 0$ (e.g., [3]). Hence $N\varphi = N\psi = 0$.

(2) We have

$$(\vartheta + \bar\vartheta)(\partial + \bar\partial)\varphi + (\partial + \bar\partial)(\vartheta + \bar\vartheta)\varphi = (\vartheta + \bar\vartheta)L\psi .$$

The left hand side of this equality is equal to $\Box\varphi + \Box\varphi = 2\,\Box\varphi$, and the right hand side is equal to $L(\vartheta + \bar{\vartheta})\psi + \sqrt{-1}(\bar{\partial}\psi - \partial\psi)$. Thus we get (2).

(3) We have $\bar{\partial}\Box\varphi = \Box\bar{\partial}\varphi$ and $\bar{\partial}L\Lambda\varphi = L\Lambda\bar{\partial}\varphi - \sqrt{-1}L\bar{\vartheta}\varphi$. Therefore it follows from (2) that

$$2\,\Box\bar{\partial}\varphi + L\Lambda\bar{\partial}\varphi = \sqrt{-1}L\bar{\vartheta}\varphi\ .$$

Analogously we have

$$2\,\Box\partial\varphi + L\Lambda\partial\varphi = -\sqrt{-1}L\vartheta\varphi .$$

From these two equalities follows that

$$2\Box(\bar{\partial}\varphi - \partial\varphi) + L\Lambda(\bar{\partial}\varphi - \partial\varphi) = \sqrt{-1}L(\bar{\vartheta}\varphi + \vartheta\varphi) = 0.$$

Hence $\bar{\partial}(\bar{\partial}\varphi - \partial\varphi) = -\bar{\partial}\partial\varphi = 0$.

(4) We have $L\bar{\partial}\psi = \bar{\partial}L\psi = \bar{\partial}(\bar{\partial}\varphi + \partial\varphi) = \bar{\partial}\partial\varphi = 0$, and hence

$$0 = \Lambda L\bar{\partial}\psi = L\Lambda\bar{\partial}\psi + (n-k-1)\bar{\partial}\psi\ .$$

Since $k \leq n-1$, this equality gives $\Lambda\bar{\partial}\psi = 0$. Since $\Lambda\Box\varphi = \Box\Lambda\varphi$, it follows from (2) that

$$2\,\Box\Lambda\varphi + \Lambda L\Lambda\varphi = 2\sqrt{-1}\Lambda\bar{\partial}\psi = 0, \quad \text{whence} \quad \Lambda\varphi = 0.$$

(5) Since $\Lambda\varphi = 0$, we have $\Box\bar{\partial}\varphi = 0$ by (2), whence

$\bar\partial\varphi = 0$. Analogously we obtain $\partial\varphi = 0$. Hence we have $L\psi = \bar\partial\varphi + \partial\varphi = 0$. Therefore it follows that $0 = \Lambda L\psi = L\Lambda\psi + (n-k)\psi$. Since $k \le n-1$, this equality gives $\psi = 0$. Thus (2) reduces to $\Box\varphi = 0$ and hence we have proved $\psi = 0$ and $\alpha = \varphi \in K^k_\Lambda(M)$.

Proof of Theorem 13.2. $\alpha \in A^{p+q}(M)$ is in $C^{p,q}(M)$ if and only if $\pi_0\alpha \in B^{p,q}(M)$ and $\pi_1\alpha \in B^{p-1,q}(M)$.

Lemma 13.5. Let $\alpha \in C^{p,q}(M)$.

(1) $\pi_0 d''\alpha = \bar\partial\pi_0\alpha - L\pi_1\alpha,$

$\pi_1 d''\alpha = -\bar\partial\,\pi_1\alpha$.

(2) $\pi_0\delta''\alpha = \vartheta\,\pi_0\alpha$,

$\pi_1\delta''\alpha = -(\Lambda\pi_0\alpha + \vartheta\,\pi_1\alpha)$.

Proof. $\bar\partial\pi_0\alpha,\ L\pi_1\alpha \in B^{p,q+1}(M)$, $\bar\partial\pi_1\alpha \in B^{p-1,q+1}(M)$, $\bar\partial\pi_0\alpha \in B^{p+1,q}(M)$ and $N\pi_0\alpha,\ \partial\pi_1\alpha \in B^{p,q}(M)$. Therefore (1) follows from the equality for $d\alpha$ in the proof of Lemma 13.3. (2) is easy from (1).

Let $\alpha = \varphi + e(\theta)\psi \in C^{p,q}(M)$. By Lemma 13.5, α is in $H^{p,q}_{(0)}(M)$ if and only if φ and ψ satisfy the equations:

$$\begin{cases} N\varphi = N\psi = 0, \\ \bar{\partial}\varphi = L\psi, \quad \bar{\partial}\psi = 0 \\ \vartheta\varphi = 0, \quad \vartheta\psi = -\Lambda\varphi \ . \end{cases}$$

To prove Theorem 13.2, it is sufficent to verify the first assertaion (1) (cf. the proof of Theorem 13.1). Clearly we have $K_{\Lambda}^{p,q}(M) \subset H_{(0)}^{p,q}(M)$.

Conversely we have $H_{(0)}^{p,q} \subset K_{\Lambda}^{p,q}(M)$ by the following

Lemma 13.6. Let $\alpha = \varphi + e(\theta)\psi \in H_{(0)}^{p,q}(M)$.

(1) $\Box\varphi + L\Lambda\varphi = -\sqrt{-1}\partial\psi$.

(2) $\bar{\partial}\partial\varphi = 0$.

(3) $\Lambda\varphi = 0$.

(4) $\psi = 0$, and $\alpha = \varphi \in K_{\Lambda}^{p,q}(M)$.

The proof of this lemma is analogous to that of Lemma 13.4 and therefore is omitted.

13.2. The groups $H_{(\lambda)}^{p,q}(M)$, $\lambda \neq 0$. For every non-zero real number λ, we define differential operators

$$T_{\lambda} : \mathcal{B}^{p,q}(M) \times \mathcal{B}^{p-1,q}(M) \to \mathcal{B}^{p,q}(M) \times \mathcal{B}^{p-1,q}(M)$$

by

$$\begin{cases} \alpha = \varphi - \dfrac{\sqrt{-1}}{\lambda}\,\partial\psi, \\ \beta = \psi - \dfrac{\sqrt{-1}}{\lambda}\,\bar{\vartheta}\varphi, \end{cases}$$

where $\varphi \in B^{p,q}(M)$, $\psi \in B^{p-1,q}(M)$ and $(\alpha, \beta) = T_\lambda(\varphi, \psi)$. In this paragraph we shall identify $C^{p,q}(M)$ with the product space $B^{p,q}(M) \times B^{p-1,q}(M)$ by the correspondence $\alpha \to (\pi_0\alpha, \pi_1\alpha)$.

Theorem 13.7. Let (p, q) be any pair of integers, and λ any real number with $|\lambda| > 1$. Then the operator T_λ maps $H^{p,q}_{(\lambda)}(M)$ injectively onto $\tilde{H}^{p,q}_{(\lambda)}(M) \times \tilde{H}^{p-1,q}_{(\lambda)}(M)$.

Corollary (cf. Naruki [23]). Let (p, q) be any pair of integers, and λ any non-zero real number. Then we have the isomorphism :

$$H^{p,q}_{(\lambda)}(M) \cong \tilde{H}^{p,q}_{(\lambda)}(m) \times \tilde{H}^{p-1,q}_{(\lambda)}(M).$$

Proof (of the corollary). The groups $H^{p,q}_{(\lambda)}(M)$, $\tilde{H}^{p,q}_{(\lambda)}(M)$ and the spaces $H^{p,q}_{(\lambda)}(M)$, $\tilde{H}^{p,q}_{(\lambda)}(M)$ are all dependent of the analytic basic field ξ chosen. Accordingly we write them exactly as $H^{p,q}_{(\lambda)}(M, \xi)$, etc. Let $\Delta^{p,q}(\xi) = \{\lambda \in \mathbb{R} \mid H^{p,q}_{(\lambda)}(M, \xi) \neq 0\}$ and $\tilde{\Delta}^{p,q}(\xi) =$

$\{ \lambda \in \mathbb{R} \mid \tilde{H}^{p,q}_{(\lambda)}(M, \xi) \neq 0 \}$. Let us now modify ξ by a positive constant ρ to consider the analytic basic field $\rho\xi$. Then we have $\Delta^{p,q}(\rho\xi) = \rho\cdot\Delta^{p,q}(\xi)$ and $\tilde{\Delta}^{p,q}(\rho\xi) = \rho\cdot\tilde{\Delta}^{p,q}(\xi)$, and

$$\mathcal{H}^{p,q}_{(\rho\lambda)}(M, \rho\xi) \cong H^{p,q}_{(\rho\lambda)}(M, \rho\xi) = H^{p,q}_{(\lambda)}(M, \xi),$$

$$\tilde{\mathcal{H}}^{p,q}_{(\rho\lambda)}(M, \rho\xi) \cong \tilde{H}^{p,q}_{(\rho\lambda)}(M, \rho\xi) = \tilde{H}^{p,q}_{(\lambda)}(M, \xi).$$

(Note that the operators $\bar{\partial} : \mathcal{B}^{p,q}(M) \to \mathcal{B}^{p,q+1}(M)$ are unchangeable under the modification.) Since both $\Delta^{p,q}(\xi)$ and $\tilde{\Delta}^{p,q}(\xi)$ are discrete subsets of $\mathbb{R}$, we can find a ρ such that $|\lambda| > 1$ for any $\lambda \in \Delta^{p,q}(\rho\xi) \cup \tilde{\Delta}^{p,q}(\rho\xi) - \{0\}$ and any pair (p, q). By Theorem 13.7, then we have, for any (p, q) and any λ,

$$\mathcal{H}^{p,q}_{(\rho\lambda)}(M,\rho\xi) \cong \tilde{\mathcal{H}}^{p,q}_{(\rho\lambda)}(M,\rho\xi) \times \tilde{\mathcal{H}}^{p-1,q}_{(\rho\lambda)}(M,\rho\xi).$$

Thus we get the corollary.

Proof of Theorem 13. 7. Let $(\varphi, \psi) \in C^{p,q}(M)$. By Lemma 13. 5 (φ, ψ) is in $\mathcal{H}^{p,q}_{(\lambda)}(M)$ if and only if (φ, ψ) satisfies the equations:

$$(13.\ 2)\begin{cases} N\varphi = \lambda\varphi, \quad N\psi = \lambda\psi, \\ \bar{\partial}\varphi = L\psi, \quad \bar{\partial}\psi = 0, \\ \vartheta\varphi = 0, \quad \vartheta\psi = -\Lambda\varphi. \end{cases}$$

Lemma 13. 8. If $(\varphi, \psi) \in H^{p,q}_{(\lambda)}(M)$, then we have $(\alpha, \beta) = T_\lambda(\varphi, \psi) \in \tilde{H}^{p,q}_{(\lambda)}(M) \times \tilde{H}^{p-1,q}_{(\lambda)}(M)$.

Proof. It is sufficient to prove the first assertion that $\alpha \in \tilde{H}^{p,q}_{(\lambda)}(M)$, because the second assertion that $\beta \in \tilde{H}^{p-1,q}_{(\lambda)}(M)$ can be similarly dealt with or rather can be derived from the first by using the dualities given by $\#_A$ and $\#_B$.

By using (13. 2) and Lemmas 12 1 $\sim$ 12. 4, we can easily obtain $N\alpha = \lambda\alpha$, $\bar{\partial}\alpha = 0$ and $\vartheta\alpha = -\frac{\sqrt{-1}}{\lambda}\Lambda\partial\alpha$. (For example,

$$\begin{aligned}\vartheta\alpha &= \vartheta\varphi - \frac{\sqrt{-1}}{\lambda}\vartheta\partial\psi = \frac{\sqrt{-1}}{\lambda}\partial\vartheta\psi = -\frac{\sqrt{-1}}{\lambda}\partial\Lambda\varphi \\ &= -\frac{\sqrt{-1}}{\lambda}(\Lambda\partial\varphi + \sqrt{-1}\vartheta\varphi) = -\frac{\sqrt{-1}}{\lambda}\Lambda\partial\alpha.)\end{aligned}$$

It remains to show $\vartheta\alpha = 0$. First of all we obtain

$$\partial\vartheta\alpha = -\frac{1}{\lambda}\partial\vartheta\alpha.$$

Since $\lambda \neq -1$, it follows that $\partial\vartheta\alpha = 0$. Furthermore we have

$$\Box\alpha + \Lambda L\alpha = -\frac{1}{\lambda}\bar{\vartheta}\partial\alpha.$$

From this equality together with $\partial\vartheta\alpha = 0$, we have

$$\Box\vartheta\alpha + \Lambda L\vartheta\alpha = 0.$$

Hence $\vartheta\alpha = 0$, proving Lemma 13. 8.

We now define a differential operator

$$U_\lambda \ : \mathcal{B}^{p,q}(M) \to \mathcal{B}^{p,q}(M)$$

by

$$U_\lambda \ = \frac{1}{\lambda^2}\,\bar{\Box}\,\varphi + \frac{1}{2\lambda^2}\,N^2\varphi + \frac{1}{2}\,\varphi, \qquad \varphi \in \mathcal{B}^{p,q}(M),$$

which is a self-adjoint, strongly elliptic operator. It is clear that the operator U_λ maps $\mathcal{B}^{p,q}(M)$ injectively onto itself. Let $\mathcal{B}^{p,q}_{(\lambda)}$ denote the subspace of $\mathcal{B}^{p,q}(M)$ defined by

$$\mathcal{B}^{p,q}_{(\lambda)} = \{\, \varphi \in \mathcal{B}^{p,q}(M) \mid N\varphi = \lambda\varphi \,\}.$$

Then it is easy to see that $U_\lambda \mathcal{B}^{p,q}_{(\lambda)} = \mathcal{B}^{p,q}_{(\lambda)}$ and

$$U_\lambda\varphi = \frac{1}{\lambda^2}\,\bar{\Box}\varphi + \varphi, \qquad \varphi \in \mathcal{B}^{p,q}_{(\lambda)} .$$

An easy calculation gives the following

Lemma 13. 9. Take any $(\varphi, \psi) \in \mathcal{B}^{p,q}_{(\lambda)} \times \mathcal{B}^{p-1,q}_{(\lambda)}$ and put $(\alpha, \beta) = T_\lambda(\varphi, \psi)$. Then we have :

$$U_\lambda\varphi = \alpha + \frac{1}{\lambda^2}\bar{\vartheta}\partial\alpha + \frac{\sqrt{-1}}{\lambda}\,\partial\beta \ ,$$

$$U_\lambda\,\psi = \beta + \frac{1}{\lambda^2}\partial\bar{\vartheta}\,\beta + \frac{\sqrt{-1}}{\lambda}\,\bar{\vartheta}\alpha.$$

Lemma 13.10. The operator T_λ maps $\mathcal{B}^{p,q}_{(\lambda)} \times \mathcal{B}^{p-1,q}_{(\lambda)}$ injectively onto itself.

Proof. Let $(\alpha, \beta) \in \mathcal{B}^{p,q}_{(\lambda)} \times \mathcal{B}^{p-1,q}_{(\lambda)}$. Then there is a unique $(\varphi, \psi) \in \mathcal{B}^{p,q}_{(\lambda)} \times \mathcal{B}^{p-1,q}_{(\lambda)}$ satisfying the equalities in Lemma 13.9. We have

$$U_\lambda(\varphi - \alpha) = -\frac{1}{\lambda^2}\partial\overline{\vartheta}\alpha + \frac{\sqrt{-1}}{\lambda}\partial\beta = U_\lambda\left(\frac{\sqrt{-1}}{\lambda}\partial\psi\right),$$

whence $\varphi - \alpha = \frac{\sqrt{-1}}{\lambda}\partial\psi$. In the same way we get $\psi - \beta = \frac{\sqrt{-1}}{\lambda}\overline{\vartheta}\varphi$. We have thus proved T_λ to be surjective. That T_λ is injective, is clear from Lemma 13. 9.

Lemma 13.11. Let $(\varphi, \psi) \in \mathcal{B}^{p,q}_{(\lambda)} \times \mathcal{B}^{p-1,q}_{(\lambda)}$. If $(\alpha, \beta) = T_\lambda(\varphi, \psi) \in \tilde{H}^{p,q}_{(\lambda)}(M) \times \tilde{H}^{p-1,q}_{(\lambda)}(M)$, then we have $(\varphi, \psi) \in H^{p,q}_{(\lambda)}(M)$.

Proof. First we have

$$\vartheta\varphi = -\frac{1}{\lambda^2}\partial\overline{\vartheta}\vartheta\varphi - \frac{1}{\lambda}\vartheta\varphi .$$

Since $1 + \frac{1}{\lambda} > 0$, it follows that $\vartheta\varphi = 0$. Furthermore we can easily show $\vartheta\psi = -\Lambda\varphi$. In the same way we get $\overline{\partial}\varphi = L\psi$ and $\overline{\partial}\psi = 0$, proving Lemma 13.11.

Now Theorem 13. 7 follows immediately from Lemmas 13. 8, 13.10 and 13.11.

13. 3. The groups $H^{k-1,1}_{*,(0)}(M)$. The main aim of this paragraph is to prove the following

Proposition 13.12. For any k, we have :

$$\begin{aligned} H^{k-1,1}_{*,(0)}(M) &= K^{k,0}_{\Lambda}(M) \oplus e(\theta)K^{k-1,0}_{L}(M) \\ &\oplus K^{k-1,1}_{\Lambda}(M) \oplus e(\theta)K^{k-2,1}_{L}(M). \end{aligned}$$

Corollary. For any k, we have :

$$H^{k-1,1}_{*,(0)}(M) = H^{k,0}_{(0)}(M) \oplus H^{k-1,1}_{(0)}(M).$$

This fact is clear from Theorem 13. 2 and Propostion 13.12.

Consider the operators $A : C^{p,q}(M) \to C^{p+2,q-1}(M)$, $d' : C^{p,q}(M) \to C^{p+1,q}(M)$ and $\delta' : C^{p+1,q}(M) \to C^{p,q}(M)$ which were defined in 8. 1. We have $T(X, Y) = 0$ for all $X, Y \in \hat{T}(M)_x$ (see 11. 1). Therefore from the formula for $A\varphi$ given in 8. 1, we see that the operator A vanishes and hence

$$d\varphi = d'\varphi + d''\varphi, \quad \varphi \in A^{p,q}(M).$$

(Accordingly the collection $\{C^{p,q}(M), d', d''\}$ gives a double complex in a proper sense.)

Lemma 13.13. Let $\alpha \in C^{p,q}(M)$.

(1) $\pi_0 d'\alpha = \partial\pi_0\alpha,$

$\pi_1 d'\alpha = -\sqrt{-1}\, N\pi_0\alpha - \partial\pi_1\alpha.$

(2) $\pi_0\delta'\alpha = \bar{\vartheta}\,\pi_0\alpha + \sqrt{-1}\, N\pi_1\alpha,$

$\pi_1\delta'\alpha = -\bar{\vartheta}\pi_1\alpha.$

This is easy from the proof of Lemmas 13. 3 and 13. 5.

We are now in a position to prove Proposition 13.12. Let $\alpha = \alpha_0 + \alpha_1 \in A^{k-1,1}(M)$, where $\alpha_0 \in C^{k,0}(M)$ and $\alpha_1 \in C^{k-1,1}(M)$. And let $\alpha_0 = \varphi_0 + e(\theta)\psi_0$ and $\alpha_1 = \varphi_1 + e(\theta)\psi_1$, where $\varphi_0 \in B^{k,0}(M)$, $\psi_0 \in B^{k-1,0}(M)$, $\varphi_1 \in B^{k-1,1}(M)$ and $\psi_1 \in B^{k-2,1}(M)$. Then we see from Lemmas 13. 5 and 13.13 that α is in $H^{k-1,1}_{*,(0)}(M)$ if and only if φ_0, ψ_0, φ_1 and ψ_1 satisfy the equations :

$$(13.\ 3)\quad \begin{cases} N\varphi_0 = N\psi_0 = N\varphi_1 = N\psi_1 = 0, \\ \partial\varphi_0 = \partial\psi_0 = 0, \\ \bar{\partial}\varphi_0 - L\psi_0 + \partial\varphi_1 = 0, \quad \bar{\partial}\psi_0 + \partial\psi_1 = 0, \\ \bar{\partial}\varphi_1 - L\psi_1 = 0, \quad \bar{\partial}\psi_1 = 0, \\ \bar{\vartheta}\varphi_0 + \vartheta\varphi_1 = 0, \quad \bar{\vartheta}\psi_0 + \Lambda\varphi_1 + \vartheta\psi_1 = 0\ . \end{cases}$$

First of all it is clear that $\alpha \in H^{k-1,1}_{*,(0)}(M)$ if $\varphi_0 \in K^{k,0}_{\Lambda}(M)$,

$\psi_0 \in K_L^{k-1,0}(M)$, $\varphi_1 \in K_\Lambda^{k-1,1}(M)$ and $\psi_1 \in K_L^{k-2,1}(M)$.

Conversely let $\alpha \in H_{*,(0)}^{k-1,1}(M)$. By using (13. 3) and Lemmas 12. 1 $\sim$ 12. 4, we have $(\bar\partial\psi_0, \varphi_1) = 0$ and

$$\square\,\varphi_1 + \bar{\vartheta}\partial\varphi_1 + L\Lambda\varphi_1 = 2\sqrt{-1}\;\bar\partial\psi_0,$$

whence

$$(\square\,\varphi_1, \varphi_1) + (\partial\varphi_1, \partial\varphi_1) + (\Lambda\varphi_1, \Lambda\varphi_1) = 0.$$

It follows that $\varphi_1 \in K_\Lambda^{k-1,1}(M)$ and hence that $\partial\varphi_0 = \bar{\vartheta}\varphi_0 = 0$, i.e., $\varphi_0 \in K_\Lambda^{k,0}(M)$. Furtheremore we easily obtain

$$\vartheta\bar\partial\psi_0 + \partial\bar{\vartheta}\psi_0 = 0,$$

meaning $\bar\partial\psi_0 = \bar{\vartheta}\psi_0 = 0$. Thus we get $\partial\psi_0 = \bar{\vartheta}\psi_0 = L\psi_0 = 0$ and $\bar\partial\psi_1 = \vartheta\psi_1 = L\psi_1 = 0$, i.e., $\psi_0 \in K_L^{k-1,0}(M)$ and $\psi_1 \in K_L^{k-2,1}(M)$. We have thereby proved Propostion 13.12.

13. 4. The groups $H_0^k(M)$. Since $NS^k(M) \subset S^k(M)$, the operator N operates on the cohomology group $H_0^k(M)$. We assert that the operation on $H_0^k(M)$ is trivial. Indeed, let $\varphi \in S^k(M)$ be such that $d\varphi = 0$. Then $L_\xi\varphi = \xi \lrcorner d\varphi + d(\xi \lrcorner \varphi) = d(\xi \lrcorner \varphi)$. ξ being analytic, we have $\xi \lrcorner \varphi \in S^{k-1}(M)$. Hence $N\varphi \in dS^{k-1}(M)$, proving

our assertion.

Consider the exact sequence

$$0 \to H_0^k(M) \to H_*^{k-1,1}(M) \to H^{k-1,1}(M)$$

(see Propostion 1. 2). By the remark above, we see that this exact sequence induces the exact sequences

$$0 \to H_0^k(M) \to H_{*,(0)}^{k-1,1}(M) \to H_{(0)}^{k-1,1}(M),$$

$$0 \to H_{*,(\lambda)}^{k-1,1}(M) \to H_{(\lambda)}^{k-1,1}(M) \qquad (\lambda \neq 0).$$

Since $\mathcal{H}_{*,(0)}^{k-1,1}(M) = \mathcal{H}_{(0)}^{k,0}(M) \oplus \mathcal{H}_{(0)}^{k-1,1}(M)$ by Corollary to Proposition 13.12 and since the map $H_{*,(0)}^{k-1,1}(M) \to H_{(0)}^{k-1,1}(M)$ is induced from the orthogonal projection $\mathcal{H}_{*,(0)}^{k-1,1}(M) \to \mathcal{H}_{(0)}^{k-1,1}(M)$, we have the isomorphism

$$H_0^k(M) \cong \mathcal{H}_{(0)}^{k,0}(M) \cong H_{(0)}^{k,0}(M)$$

and the exact sequence

$$0 \to H_0^k(M) \to H_{*,(0)}^{k-1,1}(M) \to H_{(0)}^{k-1,1}(M) \to 0.$$

Therefore we have proved

Theorem 13.14. For any k, we have :

(1) $H_0^k(M) \cong H_{(0)}^{k,0}(M)$.

(2) $0 \to H_0^k(M) \to H_{*,(0)}^{k-1,1}(M) \to H_{(0)}^{k-1,1}(M) \to 0$ (exact).

Corollary. $H_0^{n-1}(M) \cong H_0^n(M)$.

Proof. $H_0^{n-1}(M) \cong \mathcal{H}_{(0)}^{n-1,0}(M) \cong \mathcal{K}_\Lambda^{n-1,0}(M)$, $H_0^n(M) \cong \mathcal{H}_{(0)}^{n,0}(M) \cong \mathcal{K}_L^{n-1,0}(M)$, and $\mathcal{K}_\Lambda^{n-1,0}(M) = \mathcal{K}_L^{n-1,0}(M)$.

This corollary is interesting in connection with Naruki's formula, (N. 1), for the Milnor number μ.

13. 5. The groups $H_{*,(\lambda)}^{k-1,1}(M)$, $\lambda \neq 0$. In this paragraph we state the following

Proposition 13.15. Let k be any integer, and λ any real number with $|\lambda| > 1$. Then $\mathcal{H}_{*,(\lambda)}^{k-1,1}(M)$ is contained in $\mathcal{H}_{(\lambda)}^{k-1,1}(M)$, and

$$T_\lambda(\mathcal{H}_{*,(\lambda)}^{k-1,1}(M)) = 0 \times \tilde{\mathcal{H}}_{(\lambda)}^{k-2,1}(M),$$

where T_λ is the injective operator of $\mathcal{B}_{(\lambda)}^{k-1,1} \times \mathcal{B}_{(\lambda)}^{k-2,1}$ onto itself given in 13. 2.

Corollary. For any integer k and any real number λ, we have the isomorphism :

$$H_{*,(\lambda)}^{k-1,1}(M) \cong \tilde{H}_{(\lambda)}^{k-2,1}(M).$$

The proof of these facts is left to the readers as an exercise.

Appendix

Linear differential systems

In this appendix the differentiability will always mean that of class C^∞ unless otherwise stated.

Let Φ be a sheaf of vector spaces on a manifold M. For each $p \in M$, Φ_p will denote the stalk of Φ at p. $\Gamma(\Phi)$ will denote the space of cross sections of Φ, and $\Gamma_0(\Phi)$ the space of cross sections with compact support of Φ.

Let X be a vector field with compact support on a manifold M. As is well known, X generates a global 1-parameter group $\{\varphi_t\}$ of transformations of M. The transformation φ_1 which is usually called the exponential map generated by X, will be denoted by e^X. We have $\varphi_t = e^{tX}$. For any differentiable function f on M, the function $f \circ e^{tX}$ is expanded to a formal power series in t as follows :

$$f \circ e^{tX} \sim \sum_m \frac{t^m}{m!} X^m f.$$

1. Linear differential systems

Let M be a manifold. $\mathcal{O}$ denotes the sheaf of local differentiable functions on M, being a sheaf of rings. Given a subbundle P of the tangent bundle $T(M)$ of M, $\underline{P}$ denotes the sheaf of local cross sections of P. The sheaf $\underline{T(M)}$ is at the same time an $\mathcal{O}$-module and a sheaf of Lie algebras with respect to the usual bracket operation $[\ ,\ ]$.

By a linear differential system or simply a differentiable system on M, we mean an $\mathcal{O}$-submodule of the $\mathcal{O}$-module $\underline{T(M)}$.

Let Φ be a differential system on M. For any integer $\ell \geq 1$, we define a subsheaf Φ^{ℓ} of $\underline{T(M)}$ inductively by $\Phi^1 = \Phi$ and

$$\Phi^{\ell} = [\Phi, \Phi^{\ell-1}] + \Phi^{\ell-1},$$

i.e., each stalk Φ^{ℓ}_p of Φ^{ℓ} is defined to be the subspace of $\underline{T(M)}_p$ spanned by the elements of the form

$$\mathrm{ad}X_1 \ldots \mathrm{ad}X_{m-1}X_m,$$

where $X_1, \ldots, X_m \in \Phi_p$ and $1 \leq m \leq \ell$. Clearly we have

$$\Phi = \Phi^1 \subset \ldots \subset \Phi^{\ell} \subset \ldots$$

and

$$[\Phi^k, \Phi^\ell] \subset \Phi^{k+\ell}.$$

The union $\Phi' = \bigcup_\ell \Phi^\ell$ may be characterized as the subsheaf of the sheaf $\underline{T(M)}$ of Lie algebras generated by Φ, i.e., each stalk Φ'_p of Φ' is the subalgebra of $\underline{T(M)}_p$ generated by Φ_p. Note that Φ^ℓ and Φ' are all differential systems.

For each $p \in M$, we define a subspace $V(\Phi)_p$ of the tangent space $T(M)_p$ by

$$V(\Phi)_p = \{ X_p \mid X \in \Phi_p \}.$$

Then the union $V(\Phi) = \bigcup_p V(\Phi)_p$ forms a subbundle with singularities of $T(M)$. The differential system Φ is said to be regular if $\dim V(\Phi)_p$ is constant for all $p \in M$. If Φ is regular, then $V(\Phi)$ is a subbundle (without singularities), and $\Phi = \underline{V(\Phi)}$. Conversely if P is a subbundle of $T(M)$, then $\underline{P}$ is a regular differential system and $P = V(\underline{P})$.

We say that a differentiable curve $u(t)$, $a \leq t \leq b$, in M is an integral curve of the differential system Φ if the tangent vector

$\frac{du}{dt}(t)$ to the curve u at any time t is in the subspace $V(\Phi)_{u(t)}$ of $T(M)_{u(t)}$.

2. The transformations $\varphi(h)$

Let Φ be a differential system on a manifold M of dimension n. Assume that, for some point $p_0 \in M$, we have $\Phi'_{p_0} = T(M)_{p_0}$ or equivalently $V(\Phi')_{p_0} = T(M)_{p_0}$. Since

$$V(\Phi)_{p_0} = V(\Phi^1)_{p_0} \subset \dots \subset V(\Phi^\ell)_{p_0} \subset \dots$$

and $V(\Phi')_{p_0} = \bigcup_\ell V(\Phi^\ell)_{p_0}$, there is an integer k such that $V(\Phi^k)_{p_0} = T(M)_{p_0}$. Putting $n_\ell = \dim V(\Phi^\ell)_{p_0}$, we define a function $\mu(h)$ on $\mathbb{R}^n$ by

$$\mu(h) = \sum_{\ell=1}^{k} \sum_{i=n_{\ell-1}+1}^{n_\ell} |h_i|^{\frac{1}{\ell}}, \quad h = (h_1, \dots, h_n) \in \mathbb{R}^n.$$

Clearly we have

$$\mu(h) = O(|h|^{\frac{1}{k}}) \qquad \text{at } h = 0.$$

These being prepared, we shall prove the following

Proposition 1. To every $h \in \mathbb{R}^n$ there is associated a diffeomorphism $\varphi(h)$ of M so that the correspondence $h \to \varphi(h)$ has the following properties :

(1) $\varphi(h)p$ is C^1 differentiable in the two variables $h \in \mathbb{R}^n$ and $p \in M$.

(2) $\varphi(0) = 1$, the identity transformation of M, and the map $F : \mathbb{R}^n \times M \ni (h, p) \to (p, \varphi(h)p) \in M \times M$ gives a C^1 homeomorphism of a neighborhood of $(0, p_0)$ onto a neighborhood of (p_0, p_0).

(3) There are vector fields Z_A, $1 \le A \le N$, in $\Gamma_0(\Phi)$ and continuous functions $s_A(h)$ on $\mathbb{R}^n$ such that

$$\varphi(h) = e^{s_N(h)Z_N} \cdots e^{s_A(h)Z_A} \cdots e^{s_1(h)Z_1}$$

and such that the function $\sum_A |s_A(h)|$ is equivalent to the function $\mu(h)$ in the sense that, with suitable positive constants C_1 and C_2,

$$C_1\mu(h) \le \sum_A |s_A(h)| \le C_2\mu(h), \qquad h \in \mathbb{R}^n.$$

The proof of Proposition 1 is preceded by a general consideration on the exponential maps. For any real number t and any vector fields $Z_1, \ldots, Z_\ell$ in $\Gamma_0(\underline{T(M)})$, we define a transformation $\theta^{(\ell)}(t; Z_1, \ldots, Z_\ell)$ of M inductively by $\theta^{(1)}(t; Z_1) = e^{tZ_1}$ and $\theta^{(\ell)}(t; Z_1, \ldots, Z_\ell)$ $= \theta^{(\ell-1)}(t; Z_2, \ldots, Z_\ell)^{-1}\, e^{-tZ_1}\, \theta^{(\ell-1)}(t; Z_2, \ldots, Z_\ell)\, e^{tZ_1}.$

If we put $N_\ell = 3 \times 2^{\ell-1} - 2$, we see that $\theta^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)$ is a product of N_ℓ transformations of the form e^Z, $Z \in \Gamma_0(\underline{T(M)})$.

Using $\theta^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)$, we now define a transformation $\varphi^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)$ by

$$\varphi^{(\ell)}(t;\, Z_1, \ldots, Z_\ell) = \theta^{(\ell)}(|t|^{\frac{1}{\ell}};\, \varepsilon_t Z_1, Z_2, \ldots, Z_\ell), \quad \text{where}$$

ε_t stands for the sign of t, i.e., $\varepsilon_t = 1$ if $t > 0$; $= 0$ if $t = 0$; $= -1$ if $t < 0$.

Lemma 2. $\varphi^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)p$ is C^1 differentiable in the two variables t and p.

Proof. Let $x_1, \ldots, x_n$ be a coordinate system of M at any $q \in M$. Then, in a neighborhood of $(t, p) = (0, q)$, we have

$$\begin{aligned} x_i(\theta^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)p) = x_i(p) &+ t^\ell (\mathrm{ad}Z_1 \ldots \mathrm{ad}Z_{\ell-1}Z_\ell)_p x_i \\ &+ t^{\ell+1} R_i^{(\ell)}(t, p;\, Z_1, \ldots, Z_\ell), \end{aligned}$$

where the functions $R_i^{(\ell)}(t, p;\, Z_1, \ldots, Z_\ell)$ are differentiable in the two variables t and p. (This fact can be easily proved by induction on the integer ℓ.) It follows that

$$x_i(\varphi^{(\ell)}(t;\, Z_1, \ldots, Z_\ell)p) = x_i(p) + t(\mathrm{ad}Z_1 \ldots \mathrm{ad}Z_{\ell-1}Z_\ell)_p x_i$$

$$+ |t|^{1+\frac{1}{\ell}} R_i^{(\ell)}(|t|^{\frac{1}{\ell}}, p; \varepsilon_t Z_1, Z_2, \ldots, Z_\ell).$$

Lemma 2 is now clear from this equality.

Let us now construct the diffeomorphisms $\varphi(h)$. Put $n' = \sum_{\ell=1}^{k} \ell(n_\ell - n_{\ell-1})$. Since $V(\Phi^k)_{p_0} = T(M)_{p_0}$, we can find n vector fields $Y_1, \ldots, Y_n$ and n' vector fields $Z_{i,m}$ in $\Gamma_0(\Phi)$, where $n_{\ell-1} + 1 \leq i \leq n_\ell$, $1 \leq m \leq \ell$ and $1 \leq \ell \leq k$, such that

$$Y_i = \mathrm{ad}Z_{i,1} \ldots \mathrm{ad}Z_{i,\ell-1} Z_{i,\ell} \qquad \text{if } n_{\ell-1}+1 \leq i \leq n_\ell$$

and such that the n vectors $(Y_1)_{p_0}, \ldots, (Y_n)_{p_0}$ form a base of $T(M)_{p_0}$.

For any $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, we define n transformations $\varphi_i(h)$, $1 \leq i \leq n$, by

$$\varphi_i(h) = \varphi^{(\ell)}(h_i; Z_{i,1}, \ldots, Z_{i,\ell}) \quad \text{if } n_{\ell-1}+1 \leq i \leq n_\ell$$

and define a transformation $\varphi(h)$ by

$$\varphi(h) = \varphi_1(h) \ldots \varphi_n(h).$$

Clearly we have $\varphi_i(0) = \varphi(0) = 1$. Furthermore we see from Lemma 2 that $\varphi_i(h)p$ and $\varphi(h)p$ are all C^1 differentiable in the two variables h and p.

Lemma 3. The map $F : \mathbb{R}^n \times M \ni (h, p) \to (p, \varphi(h)p) \in M \times M$ gives a C^1 homeomorphism of a neighborhood of $(0, p_0)$ onto a neighborhood of (p_0, p_0).

Proof. Let $x_1, \ldots, x_n$ be a coordinate system of M at p_0. From the formula for $x_i(\varphi^{(\ell)}(t; Z_1, \ldots, Z_\ell)p)$ in the proof of Lemma 2, we see that

$$x_j(\varphi_i(h)p) = x_j(p) + (Y_i x_j)(p)h_i + O(|h|^{1+\frac{1}{\ell}})$$

$$\text{if } n_{\ell-1}+1 \leq i \leq n_\ell.$$

It follows that

$$x_j(\varphi(h)p) = x_j(p) + \sum_i (Y_i x_j)(p)h_i + O(|h|^{1+\frac{1}{k}}).$$

Since $(Y_1)_{p_0}, \ldots, (Y_n)_{p_0}$ form a base of $T(M)_{p_0}$, we have $\det((Y_i x_j)(p_0)) \neq 0$. Thus we get Lemma 3 by the implicit function theorem.

As we have remarked before, $\varphi_i(h)$, $n_{\ell-1}+1 \leq i \leq n_\ell$, are products of N_ℓ transformations of the form e^Z. Hence $\varphi(h)$ is a product of N transformations of the same form, where $N = \sum_{\ell=1}^{k} N_\ell(n_\ell - n_{\ell-1})$. More precisely $\varphi(h)$ may be expressed as

$$\varphi(h) = e^{s_N(h)Z_N} \cdots e^{s_A(h)Z_A} \cdots e^{s_1(h)Z_1},$$

where Z_A is of the form $Z_{i,m}$ and $s_A(h)$ is of the form $\pm|h_i|^{\frac{1}{\ell}}$ or $\pm\,\varepsilon_{h_i}\,|h_i|^{\frac{1}{\ell}}$. We have

$$\sum_A |s_A(h)| = \sum_\ell N_\ell \sum_{i=n_{\ell-1}+1}^{n} |h_i|^{\frac{1}{\ell}}$$

and hence

$$\mu(h) \le \sum_A |s_A(h)| \le N_k \mu(h).$$

We have thus constructed transformations $\varphi(h)$ having all the properties in Proposition 1.

The notations being as in Proposition 1, we define, for any $h \in \mathbb{R}^n$ and any t with $0 \le t \le N$, a transformation $\varphi(h, t)$ by

$$\varphi(h, t) = e^{(t-A+1)s_A(h)Z_A}\, e^{s_{A-1}(h)Z_{A-1}} \cdots e^{s_1(h)Z_1} \qquad \text{if } A-1 \le t \le A.$$

Then we have $\varphi(h, 0) = 1$ and $\varphi(h, N) = \varphi(h)$. Thus $\varphi(h, t)$, $0 \le t \le N$, give a homotopy between the identity and $\varphi(h)$. Using $\varphi(h, t)$, we now define, for any h and p, a curve $\tilde{\varphi}(h, p)$ in M by

$$\tilde{\varphi}(h, p)(t) = \varphi(h, t)p, \quad 0 \le t \le N.$$

Then we have $\tilde{\varphi}(h, p)(0) = p$ and $\tilde{\varphi}(h, p)(N) = \varphi(h)p$. We have $Z_A \in \Gamma(\Phi)$, and the curve $\tilde{\varphi}(h, p)$ restricted to the interval $[A-1, A]$ is an integral curve of Z_A. Hence $\tilde{\varphi}(h, p)$ is a piece-wise integral curve of the differential system Φ joining the two points p and $\varphi(h)p$. Therefore using (2) of Proposition 1, we have proved

Theorem 4 (cf. Chow [1]). Let Φ be a differential system on a connected manifold M. If $\Phi' = \underline{T(M)}$, then any two points p and q of M can be joined by a piece-wise integral curve of Φ which is a composition of integral curves of vector fields in $\Gamma(\Phi)$.

3. The distance functions associated with differential systems

Let Φ be a differential system on a connected paracompact manifold M. Assume that $\Phi' = \underline{T(M)}$.

Let g be a Riemannian metric on M. Given a differentiable curve $u(t)$, $a \leq t \leq b$, in M, we denote by $L(u)$ the length of u

w.r.t. g. We denote by $d(p, q)$, $p, q \in M$, the distance function associated with g, i.e., $d(p, q)$ = the infimum of the lengths $L(u)$ of all piece-wise differentiable curves joining p and q.

Taking account of Theorem 4, we now define a new distance function $\rho(p, q)$, $p, q \in M$, as follows : $\rho(p, q)$ = the infimum of the lengths $L(u)$ of all piece-wise integral curves u of Φ joining p and q. Clearly we have

$$d(p, q) \leq \rho(p, q).$$

It is now easy to see that ρ becomes really a distance function.

The notations being as above, we have

Theorem 5. Let $p_0 \in M$ and let k be an integer with $\Phi^k_{p_0} = \underline{T(M)}_{p_0}$. Then there is a neighborhood V of p_0 such that

$$\rho(p, q) \leq C\, d(p, q)^{\frac{1}{k}}, \quad p, q \in V.$$

Let us apply Proposition 1 to the pair $\{\Phi, p_0\}$.

Lemma 6.

$$\rho(p, \varphi(h)p) \leq C\, \mu(h), \quad h \in \mathbb{R}^n, \quad p \in M.$$

Proof. Since the support of Z_A is compact, we can find a constant C' such that

$$|(Z_A)_p| \leq C', \qquad p \in M, \quad 1 \leq A \leq N,$$

where $|X|$ denotes the norm of a vector X w.r.t. g. Let $h \in \mathbb{R}^n$ and $p \in M$. Putting $u(t) = \tilde{\varphi}(h, p)(t)$, we have

$$\frac{du}{dt}(t) = s_A(h)(Z_A)_{u(t)} \qquad \text{if} \quad A-1 \leq t \leq A,$$

whence

$$\left|\frac{du}{dt}(t)\right| \leq C' \sum_A |s_A(h)|, \quad 0 \leq t \leq N.$$

It follows that $L(\tilde{\varphi}(h, p)) \leq NC' \sum_A |s_A(h)|$. Since $\rho(p, \varphi(h)p) \leq L(\tilde{\varphi}(h, p))$ and $\sum_A |s_A(h)| \leq C''\mu(h)$ ((3) of Proposition 1), Lemma 6 follows.

Remark. Suppose that Φ is regular and that $V(\Phi)$ is a standard differential system in the sense of Tanaka [30]. Then we have $\mu(h) = O(\rho(p_0, \varphi(h)p_0))$, implying that the estimation in Lemma 6 is best possible in a sense.

Proof of Theorem 5. Since $\mu(h) = O(|h|^{\frac{1}{k}})$, there is a positive number δ_0 such that

$$\mu(h) \leq C_0 |h|^{\frac{1}{k}}, \qquad |h| < \delta_0 .$$

By (2) of Proposition 1 there are a positive number δ ($< \delta_0$) and a neighborhood U of p_0 such that the map F gives a C^1 homeomorphism of $U_\delta = U \times \{ h \mid |h| < \delta \}$ onto a neighborhood W of (p_0, p_0). The inverse map $(F|U_\delta)^{-1}$ of $F|U_\delta$ may be expressed as $(F|U_\delta)^{-1}(p, q) = (h(p, q), p)$. Again by (2) of Proposition 1, then we can find a neighborhood V of p_0 such that $V \times V \subset W$ and

$$C_1 \, d(p, q) \le |h(p, q)| \le C_2 \, d(p, q), \qquad p, q \in V.$$

Therefore using Lemma 6, we get an inequality of the form in Theorem 5.

Corollary (to Theorem 5). Let Φ and g be as in Theorem 5. Assume further that $\Phi^k = \underline{T(M)}$ with some k. Then for any compact subset K of M, we have

$$\rho(p, q) \le C \, d(p, q)^{\frac{1}{k}}, \qquad p, q \in K.$$

4. Differential systems and Hölder norms

Let Ω be a domain of $\mathbb{R}^n$. We denote by $\|f\|$ the maximum norm of a function f in $C_0^\infty(\Omega)$. Given a real number σ with $0 < \sigma \le 1$, we define the Hölder norm $|f|_\sigma$ of f by

$$|f|_\sigma = |f| + \operatorname*{Sup}_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\sigma} .$$

Theorem 7. Let $X_1, \ldots, X_r$ be vector fields on Ω. Assume that $\Phi^k = T(\Omega)$ with some k, where Φ stands for the differential system on Ω generated by $X_1, \ldots, X_r$. Then, for any compact subset K of Ω, we have

$$|f|_{\frac{1}{k}} \le C(|f| + \sum_i |X_i f|), \qquad f \in C_0^\infty(K).$$

Let $x_0 \le \Omega$. We apply Proposition 1 to the pair $\{\Phi, x_0\}$.

Lemma 8.

$$|\varphi(h)*f - f| \le C\, \mu(h) \sum_i |X_i f| ,$$

$$h \in \mathbb{R}^n, \quad f \in C_0^\infty(\Omega).$$

Proof. Let $h \in \mathbb{R}^n$ and $X \in \Omega$. Putting $u(t) = \tilde{\varphi}(h, x)(t)$, we have

$$f(\varphi(h)x) - f(x) = \int_0^N \frac{df(u(t))}{dt} dt, \qquad f \in C_0^\infty(\Omega).$$

We have

$$\frac{df(u(t))}{dt} = s_A(h)(Z_A f)(u(t)) \quad \text{if} \quad A-1 \le t \le A,$$

and Z_A may be expressed as $Z_A = \sum_i g_{iA} X_i$, where $g_{iA} \in C_0^\infty(\Omega)$.

It follows that

$$\left|\frac{df(u(t))}{dt}\right| \leq C'\mu(h) \sum_i |X_i f|(u(t)), \quad 0 \leq t \leq N.$$

Thus we get Lemma 8.

Proof of Theorem 7. From the proof of Theorem 5, we can find a neighborhood V of x_0 such that

$$\mu(h(x, y)) \leq C_0 |h(x, y)|^{\frac{1}{k}},$$

$$C_1|x - y| \leq |h(x, y)| \leq C_2|x - y|, \quad x, y \in V.$$

Therefore by Lemma 8, we obtain

$$|f(x) - f(y)| \leq C'|x - y|^{\frac{1}{k}} \sum_i \|X_i f\|,$$

$$x, y \in V, \quad f \in C_0^\infty(\Omega)$$

Now Theorem 7 can be easily derived from this fact.

5. Differential systems and Sobolev norms

Finally we shall prove the following theorem due to Hörmander [9].

Theorem 9. Let $X_1, \ldots, X_r$ be as in Theorem 7. Let σ be any real number with $0 < \sigma < \frac{1}{k}$. Then, for any compact subset K of Ω, we have

$$\|f\|_{(\sigma)}^2 \leq C(\|f\|^2 + \sum_i \|X_i f\|^2), \qquad f \in C_0^\infty(K).$$

Let $x_0 \in \Omega$. We apply Proposition 1 to the pair $\{\Phi, x_0\}$.

Lemma 10. For any $a > 0$, we have

$$\| \varphi(h)*f - f\|^2 \leq C\, \mu(h)^2 \sum_i \|X_i f\|^2,$$

$$|h| < a, \qquad f \in C_0^\infty(\Omega).$$

Proof. From the proof of Lemma 8, we see

$$|f(\varphi(h)x) - f(x)|^2 \leq C'\mu(h)^2 \sum_i \int_0^N |X_i f|^2(\varphi(h, t)x)dt.$$

We have

$$\int dx \int_0^N \sum |X_i f|^2(\varphi(h, t)x)dt \leq C'' \int |X_i f|^2(x)dx,$$

$$|h| < a, \qquad f \in C_0^\infty(\Omega).$$

Thus we obtain Lemma 10.

Proof of Theorem 9. Let V be as in the proof of Theorem 7. To prove Theorem 9, it is sufficient to deal with the case where $K \subset V$. For $\varepsilon > 0$, we define a compact subset K_ε of $\mathbb{R}^n \times \mathbb{R}^n$ by

$$K_\varepsilon = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \le \varepsilon, \quad x \in K \quad \text{or} \quad y \in K\}$$

Fix an ε with $K_\varepsilon \subset V \times V$. Then, for any $f \in C_0^\infty(K)$, we have

$$I_\varepsilon = \int_{|h|\le\varepsilon} \frac{\|\tau_h{}^*f - f\|^2}{|h|^{n+2\sigma}}\, dh$$

$$= \int\int_{K_\varepsilon} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy \le C' \int_{|h|\le C_2\varepsilon} \frac{\|\varphi(h)^*f-f\|^2}{|h|^{n+2\sigma}}\, dh,$$

where τ_h denotes the translation $\mathbb{R}^n \ni x \longmapsto x + h \in \mathbb{R}^n$. Since $0 < \sigma < \frac{1}{k}$, it follows from Lemma 10 that

$$I_\varepsilon \le C'' \sum_i \|X_i f\|^2 .$$

Furthermore we clearly have

$$\int_{|h|\ge\varepsilon} \frac{\|\tau_h{}^*f-f\|^2}{|h|^{n+2\sigma}}\, dh \le C''' \|f\|^2 .$$

Therefore using Hörmander's lemma ([7], Lemma 2.6.1, p.57), we get an equality of the form in Theorem 9.

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